Laminar Boundary Layers

Answers to problem sheet 1: Navier-Stokes equations

The Navier–Stokes equations for 2d, incompressible flow are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (2)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} - \rho g + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad (3)$$

1. Couette-Poiseuille flow

(a) For steady flow

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = 0,$$

i.e., we can ignore the time derivatives in equations 2 and 3.

The pressure gradient is applied along x and the upper plate is moving in the x direction, so the vertical velocity $v = 0$. The continuity equation 1 therefore yields,

$$\frac{\partial u}{\partial x} = 0,$$

which tells us that the horizontal velocity $u = u(y)$ only. We can therefore ignore all $x-$derivatives of u in Eqns. 2 and 3, which are now reduced to

$$-\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} = 0 \quad (4)$$

$$-\frac{\partial p}{\partial y} - \rho g = 0. \quad (5)$$

These must be solved subject to the following boundary conditions:

$$u = U, \quad v = 0 \quad \text{for} \quad y = a$$

$$u = v = 0 \quad \text{for} \quad y = 0.$$

Integrating equation 5 gives,

$$p = -\rho g y + A(x),$$

where A(x) is an integration constant, *i.e.* a function of x only. We identify $-\rho g y$ as the hydrostatic pressure and note that it does not interact with the flow: it does not appear in the equation for u since its $x-$derivative is zero.

To find the velocity profile we integrate equation 4, keeping in mind that the applied pressure gradient $\partial p/\partial x$ is constant:

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + By + C.$$

To determine the constants B and C we apply the boundary conditions:

$$u(y = 0) = 0 \Rightarrow C = 0,$$

$$u(y = a) = U \Rightarrow B = \frac{U}{a} - \frac{1}{2\mu}\frac{\partial p}{\partial x}a.$$

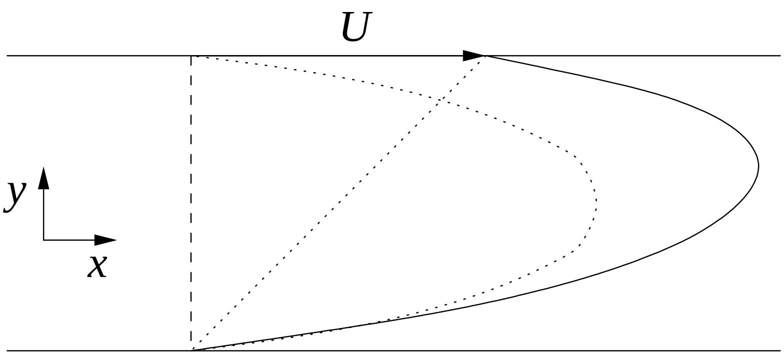
Hence, the solution is

$$u(y) = \frac{1}{2\mu}\frac{\partial p}{\partial x}y(y - a) + \frac{U}{a}y.$$

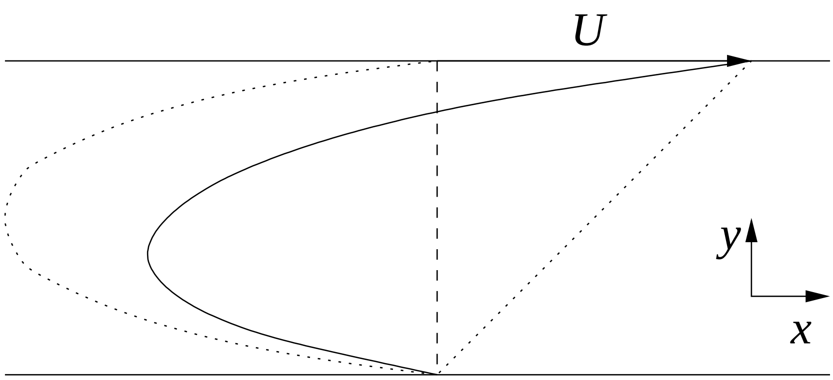
(b) The velocity profile is a superposition of two components, shown as dotted lines in the figures. The straight line Uy/a is the shear flow resulting from the movement of the upper plate $y = a$ at speed U. The parabola is due to the pressure gradient. It either points forward (favourable gradient $\partial p/\partial x < 0$) or backwards (adverse gradient $\partial p/\partial x > 0$). In the former case the maximum velocity occurs above the centre line $a/2$; in the latter case below. This can be seen from

$$\frac{\partial u}{\partial y} = 0 \Rightarrow y = \frac{a}{2} - \frac{U}{a}\frac{\mu}{\partial p/\partial x}$$

If $\partial p/\partial x < 0$



If $\partial p/\partial x > 0$



(c) Total rate of throughout:

$$\begin{aligned} Q &= \int_0^a u(y)dy \\ &= \frac{1}{2\mu}\frac{\partial p}{\partial x}\left[\frac{y^3}{3} - \frac{ay^2}{2}\right]_0^a + \frac{U}{a}\left[\frac{y^2}{2}\right]_0^a \\ &= -\frac{1}{2\mu}\frac{\partial p}{\partial x}\frac{a^3}{6} + \frac{Ua}{2} \end{aligned}$$

Mean velocity:

$$< u >= \frac{Q}{a} = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\frac{a^2}{6} + \frac{U}{2}$$

Per unit area, the drag force on the plane $y = 0$ is

$$(\Pi_{xy})_{y=0} = \left(\mu\frac{\partial u}{\partial y}\right)_{y=0} = \left(\frac{\partial p}{\partial x}(y - \frac{a}{2}) + \frac{\mu U}{a}\right)_{y=0} = \frac{\mu U}{a} - \frac{1}{2}\frac{\partial p}{\partial x}a.$$

2. Oscillatory plane

(a) The oscillating plane is infinite, so there is a translational invariance in the x direction: the velocity field does not depend on x. There is no applied pressure gradient $\partial p/\partial x$. So we can set all $x-$derivatives equal to zero in Eqns. 1 to 3.

From the continuity equation, we then get

$$\frac{\partial v}{\partial y} = 0,$$

which tells us that the vertical velocity $v = 0$ everywhere, since $v = 0$ on the boundary.

We therefore only need worry about the horizontal velocity field $u = u(y, t)$. To calculate this, we use the x-momentum equation

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}, \qquad (6)$$

where $\nu = \mu/\rho$ is the kinematic viscosity. (You might identify this as a diffusion equation.) This must be solved subject to the boundary condition

$$u(t, y = 0) = U\cos\omega t$$

We seek a solution of the form

$$u(t, y) = f(y)\cos\omega t = \mathrm{Re}(\underline{f}(y)\mathrm{e}^{i\omega t}),$$

where $\underline{f}(y)$ is a complex function. Substituting this into equation 6, we get

$$i\omega\underline{f}(y) = \nu\frac{\partial^2 \underline{f}}{\partial y^2},$$

for which the general solution is

$$\underline{f}(y) = A\mathrm{e}^{-(1+i)ky} + B\mathrm{e}^{(1+i)ky},$$

with $k = \sqrt{\omega/2\nu}$.

Hence,

$$u(y,t) = \mathrm{Re}\{A\mathrm{e}^{-ky}\mathrm{e}^{i(\omega t - ky)} + B\mathrm{e}^{ky}\mathrm{e}^{i(\omega t + ky)}\}.$$

Physically, the velocity must remain finite for all positive y no matter how far we are from the oscillating plane, so $B = 0$. Also, on the solid boundary, $u(t,0) = U\cos\omega t$, so $A = U$, and we finally get

$$u(t,y) = U\mathrm{e}^{-ky}\cos(\omega t - ky).$$

This indicates that the velocity oscillation imposed at the surface of the plane propagates towards the interior of the viscous fluid as an exponentially damped, transverse wave with velocity ω/k.

(b) The penetration depth δ of the velocity field is the distance from the plane for which the amplitude of the disturbance is decreased by a factor e^{-1}. Thus

$$\delta = \frac{1}{k} = \sqrt{\frac{2\nu}{\omega}}.$$

For water, $\nu = 10^{-6}$ m^2/s and with $\omega = 10^6$ Hz, we get $\delta = 1.4$ μm.

(c) The drag force on the plate per unit area is

$$(\Pi_{xy})_{y=0} = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu U k(\sin\omega t - \cos\omega t) = \mu U\sqrt{\frac{\omega}{2\nu}}\cos(wt - \frac{3\pi}{4}).$$

3.

$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x_i}(\rho u_i E) = E\left(\frac{\partial\rho}{\partial t} + \frac{\partial\rho u_i}{\partial x_i}\right) + \rho\left(\frac{\partial E}{\partial t} + u_i\frac{\partial E}{\partial x_i}\right) = \rho\frac{DE}{Dt},$$

because

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho u_i}{\partial x_i} = 0$$

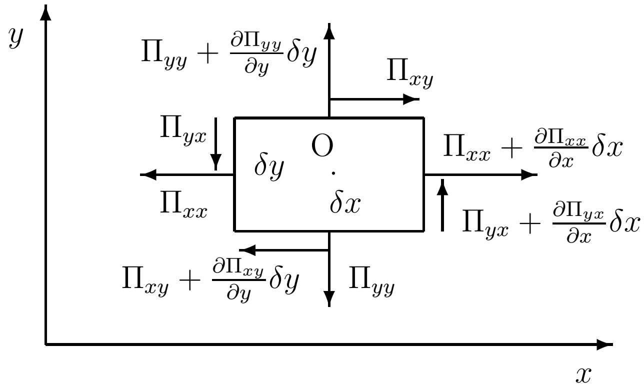
by the continuity equation.

4. Consider a two-dimensional section of a cubic element of fluid (*i.e.* a rectangle of fluid of sides δx, δy). Taking moments about the centre 0 of the rectangle, gravity forces and normal stresses (Π_{xx}, Π_{yy}) contribute nothing since their line of action passes through 0. The net moment about 0 is due to shear stresses only, and in a clockwise sense this moment is,

$$\begin{aligned}(\Pi_{xy} + \frac{\partial\Pi_{xy}}{\partial y}\delta y + \Pi_{xy})\delta x\frac{\delta y}{2} - (\Pi_{yx} + \frac{\partial\Pi_{yx}}{\partial x}\delta x + \Pi_{yx})\delta y\frac{\delta x}{2} \\ = \left(\Pi_{xy} - \Pi_{yx} + \frac{1}{2}\left\{\frac{\partial\Pi_{xy}}{\partial y}\delta y - \frac{\partial\Pi_{yx}}{\partial x}\delta x\right\}\right)\delta x\delta y.\end{aligned}$$

The moment of inertia of the cube about 0 is, per unit depth in the paper,

$$\frac{1}{12}\rho\delta x\delta y[(\delta x)^2 + (\delta y)^2].$$



Denoting the angular acceleration about 0 by a, Newton's second law for a system in rotation gives

$$\frac{1}{12}\rho\delta x\delta y[(\delta x)^2 + (\delta y)^2]a = \left(\Pi_{xy} - \Pi_{yx} + \frac{1}{2}\left\{\frac{\partial \Pi_{xy}}{\partial y}\delta y - \frac{\partial \Pi_{yx}}{\partial x}\delta x\right\}\right)\delta x\delta y.$$

Taking δx and δy to both have the same order of magnitude δ, we notice that the LHS scales as $a\delta^4$ and the RHS as $(\Pi_{xy} - \Pi_{yx})\delta^2 + O(\delta^3)$. To avoid a becoming infinite in the limit as $\delta \to 0$, the prefactor to the δ^2 term on the RHS must be zero, so we have

$$\Pi_{xy} = \Pi_{yx}.$$

This reasoning can be extended to a three-dimensional elementary cube of fluid to show that the stress tensor $[\Pi]$ is symmetric.