# Laminar Boundary Layers

## Answers to problem sheet 2: Boundary layer equations.

1.Derivation of the boundary layer equations

The 2D, incompressible boundary layer equations are derived in section 3 of the notes. Starting with the 2D N–S equations, and using the given scaled values for the variables $u$, $v$, $P$, $x$ and $y$ (as in the notes, after we choose $\delta/L = Re^{-1/2}$), we get

$$\begin{aligned}
\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} &= 0 \\
u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} &= -\frac{\partial P'}{\partial x'} + \frac{1}{Re}\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} \\
\frac{1}{Re}\left\{u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'}\right\} &= -\frac{\partial P'}{\partial y'} + \frac{1}{Re}\left\{\frac{1}{Re}\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right\}.
\end{aligned}$$

In the limit $Re \to \infty$, we can neglect all the terms containing a factor $1/Re$ or smaller. Hence, we get the non-dimensional boundary layer equations,

$$\begin{aligned}
\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} &= 0 \\
u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} &= -\frac{\partial P'}{\partial x'} + \frac{\partial^2 u'}{\partial y'^2} \\
\frac{\partial P'}{\partial y'} &= 0.
\end{aligned}$$

These must be solved subject to the boundary conditions

1. No slip and no permeation at the body surface: $u' = v' = 0$ at $y' = 0$.
2. At the exterior edge $y' \to \infty$, we have $u' \to u'_e(x')$: the slipping velocity that would be calculated at the body surface by inviscid theory.

2. Integral Momentum Equation

By integrating the momentum equation through the boundary layer, we have,

$$\begin{aligned}
\int_0^\infty \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) dy &= \int_0^\infty \left(u_e\frac{du_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2}\right) dy \\
\textit{i.e.} \quad \int_0^\infty \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - u_e\frac{du_e}{dx}\right) dy &= \nu\left[\frac{\partial u}{\partial y}\right]_0^\infty.
\end{aligned}$$

When $y \to \infty$, $u \to u_e(x)$ which is independent of $y$ so,

$$\left(\frac{\partial u}{\partial y}\right)_{y=\infty} = 0.$$

Hence,

$$\int_0^\infty \left( u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - u_e\frac{du_e}{dx} \right) dy = -\nu \left( \frac{\partial u}{\partial y} \right)_{y=0}.$$

Using the usual boundary layer scaling,

$$u' = \frac{u}{U}, \quad v' = \frac{v}{U}Re^{1/2}, \quad u'_e = \frac{u_e}{U}, \quad x' = \frac{x}{L}, \quad y' = \frac{y}{L}Re^{1/2}, \text{ with } Re = \frac{UL}{\nu},$$

we get,

$$\int_0^\infty \left( u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} - u'_e\frac{du'_e}{dx'} \right) dy' = -\left( \frac{\partial u'}{\partial y'} \right)_{y'=0} = -\frac{1}{2}c_f Re^{1/2}.$$

We have defined the skin friction coefficient as

$$\frac{1}{2}c_f Re^{1/2} = \left( \frac{\partial u'}{\partial y'} \right)_{y'=0}.$$

The left hand-side of the integral momentum equation can be re-written as,

$$\int_0^\infty \left( u'\frac{\partial u'}{\partial x'} + \frac{\partial}{\partial y'}(u'v') - u'\frac{\partial v'}{\partial y'} - u'_e\frac{du'_e}{dx'} \right) dy',$$

and the continuity equation gives,

$$\frac{\partial v'}{\partial y'} = -\frac{\partial u'}{\partial x'},$$

so we get,

$$\int_0^\infty \left( 2u'\frac{\partial u'}{\partial x'} - u'_e\frac{du'_e}{dx'} \right) dy' + [u'v']_0^\infty = -\frac{1}{2}c_f Re^{1/2}.$$

We can simplify the left hand-side term by writing,

$$\begin{aligned} 2u'\frac{\partial u'}{\partial x'} &= \frac{\partial}{\partial x'}(u'^2) \\ \text{and} \quad [u'v']_0^\infty &= \left[ -u'\int_0^{y'} \frac{\partial u'}{\partial x'}dy' \right]_0^\infty \\ &= -u'_e\int_0^\infty \frac{\partial u'}{\partial x'}dy'. \end{aligned}$$

Hence, the integral momentum equation becomes,

$$\begin{aligned} \int_0^\infty \left( -\frac{\partial}{\partial x'}(u'^2) + u'_e\frac{du'_e}{dx'} \right) dy' + u'_e\int_0^\infty \frac{\partial u'}{\partial x'}dy' &= \frac{1}{2}c_f Re^{1/2} \\ i.e. \quad \int_0^\infty \left( \frac{\partial}{\partial x'}(u'(u'_e - u')) \right) dy' + \frac{du'_e}{dx'}\int_0^\infty (u'_e - u')dy' &= \frac{1}{2}c_f Re^{1/2}. \end{aligned}$$

The momentum and displacement thicknesses are defined as,

$$\begin{aligned} \theta &= \int_0^\infty \frac{u'}{u'_e}\left( 1 - \frac{u'}{u'_e} \right) dy' \\ \delta^* &= \int_0^\infty \left( 1 - \frac{u'}{u'_e} \right) dy'. \end{aligned}$$

By substituting these expressions into the integral momentum equation, we finally get

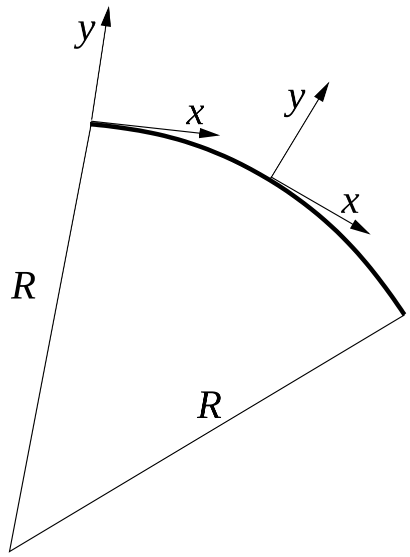
$$\frac{1}{2}c_f Re^{1/2} = \frac{d}{dx'}\left(u_e'^2\theta\right) + \delta^* u_e' \frac{du_e'}{dx'}.$$

3. <u>Boundary layer on curved surfaces</u>

For flow on a plane surface, the two-dimensional boundary layer equations gave

$$\frac{\partial P}{\partial y} = 0,$$

so that the pressure is constant through the boundary layer.




This is not strictly true for the flow over a curved surface. We adopt curvilinear coordinates, where $x$ is taken along the surface in the direction of the flow, and $y$ is normal to the flow (see sketch). In this case, a pressure gradient normal to the wall is required to balance the centrifugal forces associated with the curved flow, so we have

$$\frac{\partial P}{\partial y} = \rho \frac{u^2}{R},$$

where $R$ is the radius of curvature of the surface.

In dimensional terms, we get

$$\Delta P \sim O(\rho \frac{U^2}{R}\delta),$$

where $\delta$ is the thickness of the boundary layer at a given $x$, and $U$ the velocity scale for the $x$-direction. The latter expression can be rewritten in non-dimensional terms as

$$\frac{\Delta P}{\rho U^2} \sim O(\frac{\delta}{R}).$$

Hence, the pressure gradient through the boundary layer can be neglected if

$$\frac{\delta}{R} \ll 1.$$

3. <u>Singular limit and concept of a boundary layer</u>

(a) The exact solution of the equation is

$$w = A + B\mathrm{e}^{-x/\epsilon},$$

and the boundary conditions give

$$1 = A + B, \quad 2 = A + B\mathrm{e}^{-1/\epsilon},$$

*i.e.*

$$B = \frac{1}{\mathrm{e}^{-1/\epsilon} - 1}, \quad A = 1 - \frac{1}{\mathrm{e}^{-1/\epsilon} - 1} = \frac{\mathrm{e}^{-1/\epsilon} - 2}{\mathrm{e}^{-1/\epsilon} - 1}.$$

Hence,

$$w = \frac{1}{\mathrm{e}^{-1/\epsilon} - 1}\left(\mathrm{e}^{-1/\epsilon} - 2 + \mathrm{e}^{-x/\epsilon}\right).$$

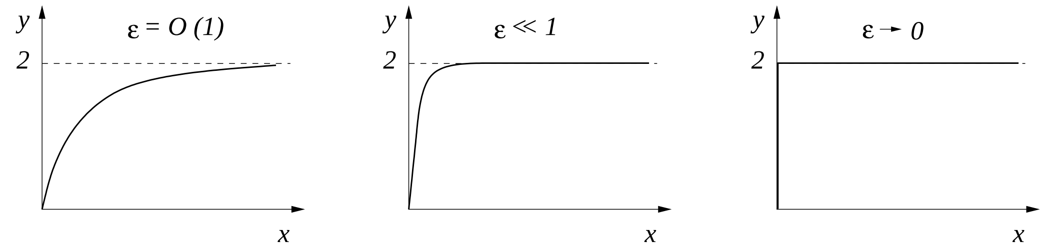
This can be simplified (for all $x$) by noting that $\exp(-1/\epsilon)$ is small when $\epsilon \ll 1$, so get

$$w \sim 2 - \mathrm{e}^{-x/\epsilon}.$$

We now examine the behaviour of this solution in the regimes $x = O(\epsilon)$ and $x = O(1)$ in turn:

(1) For $x/\epsilon = X = O(1)$ we (trivially) have $w \sim 2 - \mathrm{e}^{-X}$.
(2) For $x/\epsilon \gg 1$, we can further neglect the term in $\exp(-x/\epsilon)$ to get $w \sim -(-2) = 2$.

The solution exhibits a rapid variation near $x = 0$, *i.e.* a boundary layer. The thickness of the boundary layer is $O(\epsilon)$ (see sketch).




(b) Solution using matched expansions.

- Outer region:

$$w = w_0 + \epsilon w_1 + ... \Rightarrow \epsilon(w_0'' + \epsilon w_1'' + ..) + (w_0' + \epsilon w_1' + ...) = 0$$

To leading order, $w_0' = 0$ so $w_0 = A$. The above discussion suggests a BL at $x = 0$. So in the outer region apply the boundary condition at $x = 1 \Rightarrow A = 2$.

- Inner region:

$$w = w_0 + \epsilon w_1 + ... \text{ and } x = \epsilon X,$$

since there is a boundary layer of thickness $\epsilon$. Hence we get at leading order,

$$w_0'' + w_0' = 0,$$

with solution $w_0 = C + D\mathrm{e}^{-X}$. The boundary condition at $x = 0$ gives $C + D = 1 \Rightarrow w_0 = C + (1 - C)\mathrm{e}^{-X}$. Matching as $X \to \infty$ yields $C = 2$, *i.e.*

$$w_0 = 2 - e^{-X}.$$

These results are the same as obtained by direct solution of the equation.