Hydrodynamic stability theory

Answers to problem sheet 2. Further linear stability.

Q1. Given the basic state $\mathbf{u}_\mathrm{B} = (1 - y^2, 0)$, the NS equations reduce to

$$\partial_x p = \frac{1}{Re}\partial_y^2 u = -\frac{2}{Re}. \tag{1}$$

(Check this by considering each term of each equation in turn.) Integrating gives

$$p_\mathrm{B} = -\frac{2x}{Re} + \text{constant}. \tag{2}$$

We now linearise about the basic state. Setting $\mathbf{u} = \mathbf{u}_\mathrm{B} + \delta\tilde{\mathbf{u}}$ gives at $O(\delta)$

$$\begin{aligned} \partial_t\tilde{u} + u_\mathrm{B}\partial_x\tilde{u} + \tilde{v}u_\mathrm{B}' &= -\partial_x\tilde{p} + \frac{1}{Re}\nabla^2\tilde{u}, \\ \partial_t\tilde{v} + u_\mathrm{B}\partial_x\tilde{v} + 0 &= -\partial_y\tilde{p} + \frac{1}{Re}\nabla^2\tilde{v}, \\ \partial_x\tilde{u} + \partial_y\tilde{v} &= 0. \end{aligned} \tag{3}$$

With a normal mode form $\tilde{u} = \hat{u}(y)\exp[i\alpha(x - ct)]$, for example, we get

$$\begin{aligned} i\alpha(u_\mathrm{B} - c)\hat{u} + \hat{v}u_\mathrm{B}' &= -\hat{p}i\alpha + \frac{1}{Re}(D^2 - \alpha^2)\hat{u}, \\ i\alpha(u_\mathrm{B} - c)\hat{v} &= -\hat{p}' + \frac{1}{Re}(D^2 - \alpha^2)\hat{v}, \\ i\alpha\hat{u} + D\hat{v} &= 0. \end{aligned} \tag{4}$$

Q2. Given $\mathbf{u} = (0, V(r), 0)$ and $p = P(r)$, continuity and the axial equation are trivially satisfied. (Again, check this by examining each term in turn.) The radial equation gives

$$-\frac{V^2}{r} = -P'. \tag{5}$$

The azimuthal equation gives

$$\begin{aligned} \frac{1}{Re}\left[V'' + \frac{1}{r}V' - \frac{V}{r^2}\right] &= 0, \\ \frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \frac{V}{r^2} &= 0, \\ \frac{d}{dr}\left(\frac{d}{dr} + \frac{1}{r}\right)V &= 0. \end{aligned} \tag{6}$$

We now use this to deduce possible forms for $V(r)$. Integrating once gives

$$\left(\frac{d}{dr} + \frac{1}{r}\right) V = A \quad \text{with} \quad A = \text{const.} \tag{7}$$

Therefore

$$\frac{1}{r}\frac{d}{dr}(rV) = A. \tag{8}$$

Multiplying across by r and integrating again gives

$$rV = \frac{Ar^2}{2} + B \quad \text{with} \quad B = \text{const.} \tag{9}$$

So finally we have

$$V = \frac{Ar}{2} + \frac{B}{r}. \tag{10}$$

The perturbation equations are

$$\begin{aligned}
\partial_r \tilde{u} + \frac{\tilde{u}}{r} + \frac{1}{r}\partial_\theta \tilde{v} + \partial_z \tilde{w} &= 0, \\
\partial_t \tilde{u} + \frac{V}{r}\partial_\theta \tilde{u} - \frac{2V\tilde{v}}{r} &= -\partial_r \tilde{p} + \frac{1}{Re}\left[\Delta\tilde{u} - \frac{\tilde{u}}{r^2} - \frac{2}{r^2}\partial_\theta \tilde{v}\right] \\
\partial_t \tilde{v} + \tilde{u}V' + \frac{V}{r}\partial_\theta \tilde{v} + \frac{\tilde{u}V}{r} &= -\frac{1}{r}\partial_\theta \tilde{p} + \frac{1}{Re}\left[\Delta\tilde{v} - \frac{\tilde{v}}{r^2} + \frac{2\partial_\theta \tilde{u}}{r^2}\right] \\
\partial_t \tilde{w} + \frac{V}{r}\partial_\theta \tilde{w} &= -\partial_z \tilde{p} + \frac{1}{Re}\left[\Delta\tilde{w}\right].
\end{aligned} \tag{11}$$

Q3. The given form of solution is

$$f(x,y) = 2\cos(lx\sqrt{3})\cos(ly) + \cos(2ly). \tag{12}$$

The partial derivatives on the left hand side (LHS) are thus

$$\begin{aligned}
\partial_x^2 f &= -6l^2\cos(lx\sqrt{3})\cos(ly), \\
\partial_y^2 f &= -2l^2\cos(lx\sqrt{3})\cos(ly) - 4l^2\cos(2ly).
\end{aligned} \tag{13}$$

So the LHS is

$$\begin{aligned}
\partial_x^2 f + \partial_y^2 f &= -8l^2\cos(lx\sqrt{3})\cos(ly) - 4l^2\cos(2ly) \\
&= -4l^2\left[2\cos(lx\sqrt{3})\cos(ly) + \cos(2ly)\right] \\
&= -4l^2 f(x,y).
\end{aligned} \tag{14}$$

Therefore we require $-4l^2 = -a^2$ and so $a = 2l$.
Using the relation $2\cos(A)\cos(B) = \cos(A-B) + \cos(A+B)$ in (12), we get

$$\begin{aligned}
f(x,y) &= \cos\left(l(x\sqrt{3} - y)\right) + \cos\left(l(x\sqrt{3} + y)\right) + \cos(2ly) \\
&= \cos\left(\frac{a}{2}(x\sqrt{3} - y)\right) + \cos\left(\frac{a}{2}(x\sqrt{3} + y)\right) + \cos(ay).
\end{aligned} \tag{15}$$

Thererfore we have

$$
\begin{aligned}
f\left(x+\frac{4m\pi}{a\sqrt{3}}, y+\frac{4n\pi}{a}\right) &= \cos\left(\frac{a}{2}(x\sqrt{3}-y)+2m\pi-2n\pi\right) \\
&\quad +\cos\left(\frac{a}{2}(x\sqrt{3}+y)+2m\pi+2n\pi\right) \\
&\quad +\cos(ay+4n\pi) \\
&= f(x,y). \qquad (16)
\end{aligned}
$$

The last step follows because $2(m-n)\pi$, $2(m+n)\pi$ and $4n\pi$ additions leave the cosine function unchanged.

We now proceed to the polar transformation. Writing $f(x,t)=F(r,\theta)$ with $x=r\cos\theta, y=r\sin\theta$ in (12) we get

$$
F=\cos\left(\frac{a}{2}r(\sqrt{3}\cos\theta-\sin\theta)\right)+\cos\left(\frac{a}{2}r(\sqrt{3}\cos\theta+\sin\theta)\right)+\cos(ar\sin\theta). \qquad (17)
$$

Consider in turn the behaviour of the arguments of each of the three cosine terms in this expression under the transformation $\theta\to\theta+\pi/3$.

- The first argument contains

$$
\begin{aligned}
\sqrt{3}\cos(\theta+\pi/3)-\sin(\theta+\pi/3) &= \sqrt{3}\left\{\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta\right\}-\left\{\frac{1}{2}\sin\theta+\frac{\sqrt{3}}{2}\cos\theta\right\} \\
&= -2\sin\theta. \qquad (18)
\end{aligned}
$$

- Similarly the second argument contains

$$
\sqrt{3}\cos(\theta+\pi/3)+\sin(\theta+\pi/3)=\sqrt{3}\cos\theta-\sin\theta. \qquad (19)
$$

- Finally the third argument contains

$$
\sin(\theta+\pi/3)=\frac{1}{2}\sin\theta+\frac{\sqrt{3}}{2}\cos\theta. \qquad (20)
$$

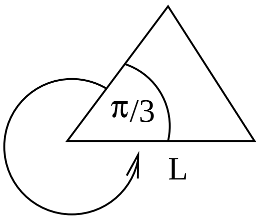
Putting all these together we get

$$
\begin{aligned}
F(r,\theta+\pi/3) &= \cos(-ar\sin\theta)+\cos\left(\frac{ar}{2}(\sqrt{3}\cos\theta-\sin\theta)\right)+\cos\left(\frac{ar}{2}\left(\sqrt{3}\cos\theta+\sin\theta\right)\right) \\
&= F(r,\theta), \qquad (21)
\end{aligned}
$$

since $\cos(-\theta)=\cos(\theta)$.

P.T.O.

So we have symmetry for rotations by $\pi/3$.



Furthermore, since

$$f\left(x + \frac{4m\pi}{a\sqrt{3}}, y\right) = f(x, y) \tag{22}$$

the "wavelength" $L = 4\pi/a\sqrt{3}$.