Hydrodynamic stability theory

Answers to problem sheet 3. Bifurcation theory.

Q1. For a steady state x_B to exist, we have $F(x_\mathrm{B}; a) = 0$. For a perturbation $x = x_\mathrm{B} + \tilde{x}$, we have

$$\frac{d\tilde{x}}{dt} = \tilde{x}\frac{dF}{dx} + \cdots, \tag{1}$$

after expanding $F(x; a)$ as a Taylor series about $x = x_\mathrm{B}$. Solving for $\tilde{x}$ provides

$$\tilde{x} = c\exp(\lambda t), \tag{2}$$

where c is constant and $\lambda = dF/dx$. Therefore the state x_B is unstable for $\lambda > 0$ and stable for $\lambda < 0$.

Q2. Steady states exist for $a > 0$ with $x_\mathrm{B} = \pm\sqrt{a}$. Using the results of Q1 above, with $F(x; a) = a - x^2$ and $dF/dx = -2x$, we see that

- For the state $x = +\sqrt{a}$, $dF/dx < 0$, and the state is linearly stable.
- For the state $x = -\sqrt{a}$, $dF/dx > 0$, and the state is linearly unstable.

When $a = 0$, we can divide by x^2 to give

$$\frac{1}{x^2}\frac{dx}{dt} = -1, \tag{3}$$

or equivalently,

$$\frac{d}{dt}\left(x^{-1}\right) = 1, \tag{4}$$

so

$$x(t) = \frac{1}{c+t}, \tag{5}$$

where $c = 1/x_0$, for $x(0) = x_0$. Thus

$$x(t) = \frac{x_0}{1 + x_0 t}. \tag{6}$$

From this, we see that:

- If $x_0 = 0$, then $x(t) = 0$ for all time.
- If $x_0 < 0$, then $|x(t)| \to \infty$ as $t \to -1/x_0$.
- If $x_0 > 0$, then $|x(t)| \to 0$ as $t \to \infty$.

Q3. On dividing by x^2, we find

$$\frac{d}{dt}\left(-x^{-1}\right) = \frac{a}{x} - b, \tag{7}$$

so

$$\frac{df}{dt} + af = b, \tag{8}$$

where $f = 1/x(t)$. Integrating provides

- If $a \neq 0$,

$$f = \frac{b}{a} + c\exp(-at), \tag{9}$$

  where for $x(0) = x_0$ (*i.e.*, $f(0) = 1/x_0$), we require $c = (1/x_0 - b/a)$.

- If $a = 0$

$$f = bt + d, \tag{10}$$

  where again the constant $d = 1/x_0$ is determined so that $x(0) = x_0$.

Hence we have

$$x(t) = \frac{ax_0}{x_0 b + (a - bx_0)\exp(-at)} \quad \text{for} \quad a \neq 0, \tag{11}$$

and

$$x = \frac{x_0}{1 + bx_0 t} \quad \text{for} \quad a = 0. \tag{12}$$

Thus we deduce the following scenarios:

- If $a = 0$ and $bx_0 > 0$, then $|x(t)| \to 0$ as $t \to \infty$.
- If $a = 0$ and $bx_0 < 0$, then $|x(t)| \to \infty$ as $t \to -1/bx_0$.
- If $a > 0$ and $bx_0 > 0$, then $|x(t)| \to a/b$ as $t \to \infty$.
- If $a < 0$ and $bx_0 > a$, then $|x(t)| \to 0$ as $t \to \infty$.
- If $a \neq 0$ and $bx_0 < \min(a, 0)$, then $|x(t)| \to \infty$ as $t \to t_0$, where

$$t_0 = \frac{1}{a}\ln\left(1 - \frac{a}{bx_0}\right). \tag{13}$$