

LAMINAR BOUNDARY LAYERS

Answers to problem sheet 4: Exact Boundary Layer Solutions

1. Suction flow

(a) The boundary conditions appropriate for this problem:

1. On the surface of the plate, *i.e.* at $y = 0$, $u = 0$ and $v = -v_0$.
2. At the exterior edge of the boundary layer, *i.e.* for $y \rightarrow \infty$, $u \rightarrow U$, the inviscid slipping velocity: $u_e(x)$ is here constant and equal to the free stream velocity U .

We seek an asymptotic solution to the equations where $u = u(y)$. Hence, the continuity equation gives,

$$\frac{\partial v}{\partial y} = 0$$

i.e. v is independent of y , so $v = -v_0$. Hence, the momentum equation reduces to

$$-v_0 \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}.$$

The solution to this equation is

$$u(y) = A + B \exp\left(-\frac{v_0}{\nu}y\right),$$

where A and B are integration constants. The boundary conditions yield

$$A + B = 0 \quad \text{and} \quad A = U.$$

Hence, the solution is

$$u(y) = U \left[1 - \exp\left(-\frac{v_0}{\nu}y\right)\right].$$

(b) The displacement thickness is given by

$$\begin{aligned} \delta^* &= \int_0^\infty \left(1 - \frac{u}{U}\right) dy \\ &= \int_0^\infty \exp\left(-\frac{v_0}{\nu}y\right) dy \\ &= \frac{\nu}{v_0}, \end{aligned}$$

and the momentum thickness is

$$\begin{aligned} \theta &= \int_0^\infty \frac{u}{U} \left(1 - \frac{u}{U}\right) dy \\ &= \int_0^\infty \exp\left(-\frac{v_0}{\nu}y\right) \left(1 - \exp\left(-\frac{v_0}{\nu}y\right)\right) dy \\ &= \int_0^\infty \left\{ \exp\left(-\frac{v_0}{\nu}y\right) - \exp\left(-\frac{2v_0}{\nu}y\right) \right\} dy \\ &= \frac{\nu}{v_0} - \frac{\nu}{2v_0}, \\ &= \frac{\nu}{2v_0}. \end{aligned}$$

Finally, the skin friction coefficient is

$$\begin{aligned} c_f &= \frac{\mu \left(\frac{\partial u}{\partial y} \right)_{y=0}}{\frac{1}{2} \rho U^2} \\ &= \frac{\mu U v_0}{\nu} \frac{2}{\rho U^2} \\ &= 2 \frac{v_0}{U}. \end{aligned}$$

(c) By substituting the solution

$$u = U \left[1 - \exp \left(-\frac{v_0}{\nu} y \right) \right]$$

and $v = -v_0$ into the full Navier–Stokes equations, you will find that the equations are satisfied.

2. Axisymmetric jet

(a) The boundary conditions on u and v are

- $u \rightarrow 0$ as $r \rightarrow \infty$ (no flow outside the jet).
- $\partial u / \partial r = v = 0$ at $r = 0$ by symmetry.

(b) Similarity solution

To determine the two unknown exponents p and q we need two conditions on the given solution $\Psi = \nu x^p g(\eta)$, $\eta = r x^{-q}$. We get one condition from each of the momentum BL equation and the integral constraint as follows.

1. In the momentum equation, the inertial and viscous terms must have the same scaling with x . We examine each of these in turn:

- Inertial terms.

First we need the scaling of the velocity component

$$u = \frac{1}{r} \frac{\partial \Psi}{\partial r}.$$

Using that fact that $\Psi = \nu x^p g(\eta)$ and $r = \eta x^q$, we find

$$u \sim x^{p-2q}$$

and so the first inertial term

$$u \frac{\partial u}{\partial x} \sim x^{2p-4q-1}.$$

The other inertial term $v \partial u / \partial r$ can be shown to scale in the same way.

- Viscous term.

Its scaling with x is found using $u \sim x^{p-2q}$ and $r \sim \eta x^q$:

$$\nu \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \sim x^{p-4q}.$$

For the inertial and viscous terms to have the same scaling, therefore, we require

$$x^{2p-4q-1} \sim x^{p-4q}$$

giving

$$2p - 4q - 1 = p - 4q$$

and hence

$$p = 1.$$

2. We are told that the integral

$$M = \int_0^\infty u^2 r dr$$

must be independent of x . Using $u \sim x^{p-2q}$ and $r \sim \eta x^q$ we get

$$M \sim x^{2p-4q} x^{2q}.$$

Hence we require

$$2(p - q) = 0$$

giving

$$q = p = 1.$$

So we now have a similarity solution of the form

$$\Psi = \nu x g(\eta) \quad \text{with} \quad \eta = \frac{r}{x}.$$

To transform the momentum BL equation from a partial differential equation in (x, r) to an ordinary differential equation in η , we make the change of variable

$$(x, r) \rightarrow \left(\xi = x, \eta = \frac{r}{x} \right),$$

noting that the reverse transformation is

$$(x = \xi, r = \xi \eta).$$

We start by calculating the partial derivatives:

$$\left(\frac{\partial}{\partial x} \right)_r = \left(\frac{\partial}{\partial \xi} \right)_\eta \left(\frac{\partial \xi}{\partial x} \right)_r + \left(\frac{\partial}{\partial \eta} \right)_\xi \left(\frac{\partial \eta}{\partial x} \right)_r = \frac{\partial}{\partial \xi} - \frac{\eta}{\xi} \frac{\partial}{\partial \eta} \quad (1)$$

and

$$\left(\frac{\partial}{\partial r} \right)_x = \left(\frac{\partial}{\partial \xi} \right)_\eta \left(\frac{\partial \xi}{\partial r} \right)_x + \left(\frac{\partial}{\partial \eta} \right)_\xi \left(\frac{\partial \eta}{\partial r} \right)_x = \frac{1}{\xi} \frac{\partial}{\partial \eta}. \quad (2)$$

Using these we find that the velocity components

$$u = \frac{1}{r} \frac{\partial \Psi}{\partial r} = \frac{\nu g'(\eta)}{\xi \eta} \quad (3)$$

and

$$v = -\frac{1}{r} \frac{\partial \Psi}{\partial x} = \frac{\nu}{\xi} \left(g' - \frac{g}{\eta} \right). \quad (4)$$

We now construct the inertial terms:

$$\begin{aligned} u \frac{\partial u}{\partial x} &= \frac{\nu g'(\eta)}{\xi \eta} \left[\frac{\partial}{\partial \xi} - \frac{\eta}{\xi} \frac{\partial}{\partial \eta} \right] \frac{\nu g'(\eta)}{\xi \eta} \\ &= \frac{\nu^2 g'(\eta)}{\xi^3 \eta} \left[-\frac{g'(\eta)}{\eta} - \eta \frac{d}{d\eta} \left(\frac{g'}{\eta} \right) \right] \end{aligned} \quad (5)$$

and

$$v \frac{\partial u}{\partial r} = \frac{\nu^2}{\xi^3} \left(g' - \frac{g}{\eta} \right) \frac{d}{d\eta} \left(\frac{g'}{\eta} \right). \quad (6)$$

Finally the viscous term

$$\frac{\nu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) = \frac{\nu^2}{\xi^3} \frac{1}{\eta} \frac{d}{d\eta} \left[\eta \frac{d}{d\eta} \left(\frac{g'(\eta)}{\eta} \right) \right]. \quad (7)$$

(I have omitted some intermediate steps in constructing each of these terms.) Putting Eqns. 5 to 7 together, we get the boundary layer momentum equation, divided across for convenience by a factor ν^2/ξ^3 :

$$\frac{g'(\eta)}{\eta} \left[-\frac{g'(\eta)}{\eta} - \eta \frac{d}{d\eta} \left(\frac{g'}{\eta} \right) \right] + \left(g' - \frac{g}{\eta} \right) \frac{d}{d\eta} \left(\frac{g'}{\eta} \right) = \frac{1}{\eta} \frac{d}{d\eta} \left[\eta \frac{d}{d\eta} \left(\frac{g'(\eta)}{\eta} \right) \right]. \quad (8)$$

Tidying up, we get

$$-\frac{g'^2}{\eta} - g \frac{d}{d\eta} \left(\frac{g'}{\eta} \right) = \frac{d}{d\eta} \left[\eta \frac{d}{d\eta} \left(\frac{g'}{\eta} \right) \right] \quad (9)$$

and finally

$$-\frac{d}{d\eta} \left(\frac{g g'}{\eta} \right) = \frac{d}{d\eta} \left(g'' - \frac{g'}{\eta} \right). \quad (10)$$

(c) We are told that

$$\eta g'' - g' + g g' = 0.$$

We recognise the LHS to be a total derivative:

$$\frac{d}{d\eta} \left[\eta g' - 2g + \frac{1}{2}g^2 \right] = 0.$$

Integrating, and using the boundary conditions to set the constant of integration to zero, we get

$$\eta g' - 2g + \frac{1}{2}g^2 = \text{const.} = 0,$$

which we write as

$$\int \frac{dg}{g(4-g)} = \frac{1}{2} \int \frac{d\eta}{\eta}.$$

Expressing the integrand on the LHS as partial fractions, integrating, and making a convenient choice for the constant of integration, we get

$$\frac{g}{4-g} = \frac{1}{4}\alpha^2\eta^2.$$

and so

$$g = \frac{\alpha^2\eta^2}{1 + \frac{1}{4}\alpha^2\eta^2}.$$

and finally (with a bit of algebra)

$$u = \frac{\nu}{\eta x} \frac{\partial g}{\partial \eta} = \frac{2\nu\alpha^2}{x \left(1 + \frac{1}{4}\alpha^2\eta^2\right)^2}.$$

To find the integration constant α , we use the constraint

$$\begin{aligned} M &= \int_0^\infty u^2 r dr \\ &= \frac{4\nu\alpha^4}{x^2} \int_0^\infty \frac{x^2 \eta d\eta}{\left(1 + \frac{1}{4}\alpha^2\eta^2\right)^4}. \end{aligned} \tag{11}$$

Changing variables $\eta \rightarrow s = 1 + \frac{1}{4}\alpha^2\eta^2$ we get

$$\begin{aligned} M &= 8\nu^2\alpha^2 \int_1^\infty \frac{ds}{s^4} \\ &= \frac{8\nu^2\alpha^2}{3}. \end{aligned} \tag{12}$$