Hydrodynamic stability theory

Answers to problem sheet 4. Further bifurcation theory.

Q1. (i) The given equations are

$$\frac{dx}{dt} = -y + (a - x^2 - y^2)x, \tag{1}$$

$$\frac{dy}{dt} = x + (a - x^2 - y^2)y. \tag{2}$$

Consider (1)+i(2):

$$\begin{aligned}\frac{d}{dt}(x+iy) &= -y + ix + (a - x^2 - y^2)(x+iy) \\ &= i(x+iy) + (a - |x+iy|^2)(x+iy). \end{aligned} \tag{3}$$

Setting $x + iy = re^{i\theta}$, we get

$$\begin{aligned}\frac{d}{dt}(re^{i\theta}) &= ire^{i\theta} + (a - r^2)re^{i\theta} \\ \frac{dr}{dt}e^{i\theta} + i\frac{d\theta}{dt}re^{i\theta} &= ire^{i\theta} + (a - r^2)re^{i\theta}. \end{aligned} \tag{4}$$

Canceling the factor $e^{i\theta}$ across, and taking the real and imaginary parts of the resulting equation, we get

$$\frac{dr}{dt} = ar - r^3, \tag{5}$$

$$\frac{d\theta}{dt} = 1 \quad \text{for} \quad r \neq 0. \tag{6}$$

(ii) Referring to page 11 in section 7 of the notes, we recognise Eqn. 5 to be of the normal form for a supercritical pitchfork bifurcation. Ignoring the phase angle θ, therefore, the bifurcation diagram or r *vs.* a will be of the form of the top sketch on page 12 of the notes, with x replaced by r on the vertical axis.

(iii) In the Argand plane $(x - y)$, r gives the distance from the origin, while θ gives the phase angle. Reinstating $\dot{\theta}$, therefore, the parabola sketched above now becomes a bowl, as in the bottom figure on page 14 of the notes. For the combined dynamics of r, θ, therefore, we have a supercritical Hopf bifurcation.

(iv) The two slices were shown in the top figure on page 14 of the notes. For $a < 0$ we have a stable focus. For $a > 0$ we have an unstable focus and a stable limit cycle.

Q2. The given equation is

$$\frac{dx}{dt} = r\ln(x) + x - 1. \tag{7}$$

Setting $x = 1$ gives $\dot{x} = 0 \ \forall\, r$. So $x = 1$ is a stationary solution for all r.

Consider small $\tilde{x} = x - 1$. Setting $x = 1 + \tilde{x}$ in (7), we get

$$\begin{aligned} \frac{d\tilde{x}}{dt} &= r\ln(1+\tilde{x}) + \tilde{x} \\ &= r\left[\tilde{x} - \frac{\tilde{x}^2}{2} + O(\tilde{x}^3)\right] + \tilde{x} \\ &= (1+r)\tilde{x} - \frac{r}{2}\tilde{x}^2 + O(\tilde{x}^3). \end{aligned} \tag{8}$$

Comparing this to Eqn. 22 on page 9 of the notes, we see that the system exhibits a transcritical bifurcation at $(1 + r) = 0$ and so $r_c = -1$.

Setting $\tilde{x} = aX$, we get

$$a\frac{dX}{dt} = (1+r)aX - \frac{r}{2}a^2X^2 + O(X^3). \tag{9}$$

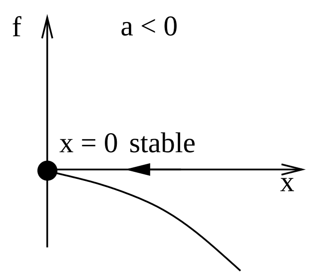
Dividing across by a, setting $a = 2/r$, $R = 1 + r$, and neglecting terms $O(X^3)$, we indeed get the normal form

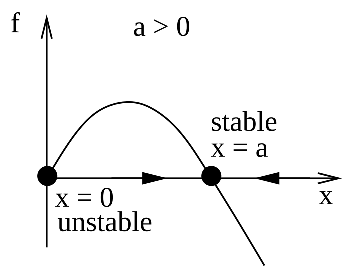
$$\frac{dX}{dt} = RX - X^2. \tag{10}$$

Q3. The given equation is

$$\frac{dx}{dt} = f(x; a) = ax - x^2. \tag{11}$$

The requested sketches are below. The bifurcation diagram is (the top half of) the figure on page 10 of the notes. Make sure you can relate the dynamics shown by the arrows on the bifurcation diagram to that shown by the arrows below.





Q4. (i) For a stationary solution $x_{n+1} = x_n$, hence $x_n = 0$ or $x_n = 1 - \lambda^{-1}$.

(ii) Consider a perturbation away from a stationary solution, so that $x_n = X_n + \epsilon_n$, where $X_n = 0$ or $X_n = 1 - \lambda^{-1}$. For the first case, if $|\epsilon_n| \ll 1$ then we can neglect the nonlinear terms to get

$$\epsilon_{n+1} = \lambda \epsilon_n, \tag{12}$$

and hence

$$\left| \frac{\epsilon_{n+1}}{\epsilon_n} \right| = |\lambda|. \tag{13}$$

Therefore the perturbation $|\epsilon_n| \to 0$ as $n \to \infty$ provided $|\lambda| < 1$. Similarly for the second stationary solution, we may linearise about $X_n = 1 - \lambda^{-1}$ to obtain

$$\epsilon_{n+1} = -\lambda(1 - \lambda^{-1})\epsilon_n + \epsilon_n. \tag{14}$$

Thus

$$\left| \frac{\epsilon_{n+1}}{\epsilon_n} \right| = |2 - \lambda|, \tag{15}$$

and the solution is stable for $1 < \lambda < 3$.

(iii) For a period-2 solution, we require $x_{n+2} = x_n$, where we have

$$x_{n+1} = \lambda x_n(1 - x_n), \tag{16}$$

and

$$x_{n+2} = \lambda x_{n+1}(1 - x_{n+1}), \tag{17}$$

so that

$$x_{n+2} = \lambda^2 x_n(1 - x_n)[1 - \lambda x_n(1 - x_n)]. \tag{18}$$

Substituting $x_{n+2} = x_n$, this becomes a fourth-order polynomial for x_n. We already know two solutions to this polynomial: the stationary solutions $x_n = 0$ and $x_n = 1 - \lambda^{-1}$ discussed above, since for those states $x_{n+2} = x_{n+1} = x_n$. We therefore now seek to factor these out.

Dividing across by x_n gives

$$x_n^3 - 2x_n^2 + \frac{1+\lambda}{\lambda} x_n + \frac{\lambda^{-2} - 1}{\lambda} = 0. \tag{19}$$

Factoring out the second stationary solution $x_n = 1 - \lambda^{-1}$ leaves a quadratic equation for the two states involved in the period-2 solution

$$\left(x_n + \frac{1-\lambda}{\lambda} \right) \left[x_n^2 - (1 + \lambda^{-1})x_n + \frac{1+\lambda}{\lambda} \right] = 0. \tag{20}$$

The quadratic finally therefore provides the two period-2 states:

$$2x_n = 1 + \frac{1}{\lambda} \pm \frac{1}{\lambda}\sqrt{\lambda^2 - 2\lambda - 3}. \tag{21}$$

(iv) Writing $x_n = X_n + \epsilon_n$, we have

$$X_{n+1} + \epsilon_{n+1} = F(X_n) + \epsilon_n F'(X_n) + \cdots, \tag{22}$$

using a Taylor series expansion of F for small perturbations ϵ_n. Then, since $X_{n+1} = F(X_n)$ by definition of the stationary points X_n, we find that

$$\epsilon_{n+1} = \epsilon_n F'(X_n). \tag{23}$$

Similarly $\epsilon_{n+2} = \epsilon_{n+1} F'(X_{n+1}) = \epsilon_n F'(X_n) F'(X_{n+1})$. Therefore, the state is linearly stable for

$$\left| \frac{\epsilon_{n+2}}{\epsilon_n} \right| = |F'(X_n) F'(X_{n+1})| < 1. \tag{24}$$