Hydrodynamic stability theory

Answers to problem sheet 5.

Q1. (1) The (lengthy!) derivation of the Stuart-Landau equation was given on pages 21 to 23 in Sec. 8 of the lecture notes. I will not write the whole thing out again here, but simply give the following additional details.

Solving (86) at the $O(\delta)$ stage of the calculation, it can be shown that λ_1 and λ_2 in (87) are given by

$$\lambda_1 = -\left(\frac{\pi k_{cm}^2}{2}\right) / \left[R_{cm}^{-1}(-4\pi^2 - 4k_{cm}^2) - 2^4 k_{cm}^4 + 4k_{cm}^2\right], \tag{1}$$

and

$$\lambda_2 = -\left(\frac{\pi k_{cm}^2}{2}\right) / \left[R_{cm}^{-1}(-4\pi^2)\right]. \tag{2}$$

Keeping track of these at the $O(\delta^{3/2})$ stage of the calculation, we find in (90)

$$c = -2k_{cm}^2 \pi \lambda_1. \tag{3}$$

and in (91)

$$d = \pi \lambda_1 k_{cm}^2 + 2\pi \lambda_2 k_{cm}^2. \tag{4}$$

So in the Stuart-Landau equation we find finally

$$\begin{aligned} \beta &= [2\lambda_1 - (\lambda_1 + 2\lambda_2)]\, k_{cm}^2 \pi \\ &= [\lambda_1 - 2\lambda_2]\, \pi k_{cm}^2, \end{aligned} \tag{5}$$

with λ_1 and λ_2 given as above.

(2) The expansion procedure is valid at any point on the neutral curve. So we can replace $k_{cm} \to k_c$, $R_{cm} \to R_c$ and follow the procedure through again. The important thing to note is that β is a function of k_c. If β changes sign at some k_c, then the bifurcation changes in nature from supercritical to subcritical (or vice versa). Note finally that the third-order amplitude equation is clearly not valid at the point where $\beta = 0$. In that case, we need to go to higher order.

Q2. (i) A trivial basic state has $U_{\mathrm{B}} = 0$. Examine its linear stability in the usual way by adding a small perturbation:

$$U = U_{\mathrm{B}} + \delta \tilde{U}. \tag{6}$$

Substituting this into the governing equation and linearising we get

$$\tilde{U}_t = \tilde{U} + \frac{1}{R}\tilde{U}_{yy}. \tag{7}$$

Assuming a normal mode solution

$$\tilde{U} = \hat{U}(y)\exp(\sigma t) \tag{8}$$

we find

$$\hat{U}(\sigma - 1) = \frac{1}{R}\hat{U}'', \tag{9}$$

subject to boundary conditions

$$\hat{U}(0) = \hat{U}(\pi) = 0. \tag{10}$$

This has the solution

$$\hat{U}(y) = \sin(ny) \tag{11}$$

with

$$\sigma = 1 - \frac{n^2}{R}, \tag{12}$$

giving loss of stability at $R = R_c = 1$, for the lowest mode $n = 1$.

(ii) Using the given expansion

$$U = \epsilon^{1/2}U_1 + \epsilon U_2 + \epsilon^{3/2}U_3 + \cdots \tag{13}$$

in which $U_1 = A(\tau)\sin(y)$, along with $R = R_c + \epsilon R_1$, we now evaluate the governing equation at successive orders $\epsilon^{1/2}, \epsilon$ and $\epsilon^{3/2}$. As a preliminary step, we note

$$U_t = \epsilon U_\tau = \epsilon^{3/2}\frac{dA}{d\tau}\sin(y) + \cdots \tag{14}$$

and

$$\frac{1}{R} = \frac{1}{R_c} - \epsilon\frac{R_1}{R_c^2} + \cdots. \tag{15}$$

Therefore at $O(\epsilon^{1/2})$ we recover the linear problem. At $O(\epsilon)$ we have

$$-U_2 = \frac{1}{R_c}U_{2\,yy} \tag{16}$$

and at $O(\epsilon^{3/2})$ we have

$$-\frac{1}{R_c}U_{3\,yy} - U_3 = -U_{1\,\tau} + U_1^3 - \frac{R_1}{R_c^2}U_{1\,yy}. \tag{17}$$

Using

$$(\sin y)^3 = \frac{3}{4}\sin y - \frac{1}{4}\sin 3y \tag{18}$$

in the term U_1^3, we find that to remove the secular terms in $\sin y$ from the RHS we require

$$-A_\tau + \frac{3}{4}A^3 + R_1 A = 0, \tag{19}$$

in which we have recalled that $R_c = 1$. Rearranging gives

$$A_\tau = R_1 A + \frac{3}{4} A^3. \tag{20}$$

Comparing this with the standard forms for each bifurcation type in the notes, we see that we have a subcritical pitchfork bifurcation.

Q3. Substituting the given cosine series into the governing equation and projecting onto the mode $e^{in\pi x}$ gives

$$\frac{1}{2}\frac{da_n}{dt} - \frac{1}{4}\sum_{-\infty}^{\infty} a_{n-m} a_m = \frac{1}{2}(-n^2\pi^2) a_n. \tag{21}$$

Multiplying across by 2 and rearranging gives

$$\frac{da_n}{dt} = -n^2\pi^2 a_n + \frac{1}{2}\sum_{-\infty}^{\infty} a_{n-m} a_m, \tag{22}$$

as required.

From this we find

$$\frac{da_0}{dt} = \frac{1}{2}\sum_{-\infty}^{\infty} a_{-m} a_m \geq \frac{1}{2} a_0^2. \tag{23}$$

Integrating gives

$$a_0 \geq \frac{2}{t_0 - t} \tag{24}$$

where $t_0 = 2/a_0(t=0)$.

If $\int_0^1 U_0(x)dx > 0$ then $a_0|_{t=0} > 0$ and so $a_0 \to \infty$ as $t \to t_0$.