Hydrodynamic stability theory

Problem sheet 1. Linear stability.

Q1. Consider the problem

$$\frac{\partial u}{\partial t} - \sin u = \frac{1}{R}\frac{\partial^2 u}{\partial y^2},$$

where R is a real parameter and

$$u(y=0,t) = u(y=\pi,t) = 0,$$

for all time, t.

Examine the linear stability of the trivial basic state $u = u_{\rm B}(y) = 0$ by seeking normal mode solutions to the linearised stability equation.

Find the eigenfunctions and eigenvalues and show that the trivial state is linearly stable only if $R \leq 1$.

Q2. Explain how one would non-dimensionalise the Benard convection problem described in the lecture notes, and therefore show that the non-dimensional version of

$$[D^2 - a^2 - s/\kappa][D^2 - a^2][D^2 - a^2 - s/\nu]\bar{w}(z) = -\frac{\alpha g a^2 \beta}{\kappa\nu}\bar{w}(z),$$

(in the notation of the lecture notes; equation 53) is

$$[D^2 - a^2 - s/P_r][D^2 - a^2][D^2 - a^2 - s]\bar{w}(z) = -a^2 R_a \bar{w}(z),$$

after suitable redefinition of D, s, a, z and definitions of P_r and R_a.

Q3. Consider the Eckhaus equation

$$R^{-1}[\Phi_{\eta\eta} + \Phi_{\xi\xi}] - \Phi_{\xi\xi\xi\xi} - \Phi_t = \Phi_\eta \Phi_{\xi\xi},$$

where R is a real parameter and $\Phi(\eta=0,\xi,t) = 0$, $\Phi(\eta=1,\xi,t) = 1$.

By considering the linear stability of the basic state $\Phi_B(\eta,\xi,t) \equiv \eta$, show that the eigenrelation is

$$n^2\pi^2 = R(k^2 - k^4 - \sigma) - k^2,$$

where the ξ and t dependencies of the perturbation to Φ take the form $\cos(k\xi)e^{\sigma t}$ and $n = 1, 2, 3, ...$

Hence show that the basic state Φ_B first becomes unstable for

$$R > R_{cm} = \frac{\pi^2 + k_{cm}^2}{k_{cm}^2 - k_{cm}^4},$$

where

$$k_{cm}^2 = \pi^2[\sqrt{(1+\pi^{-2})} - 1].$$