LAMINAR BOUNDARY LAYERS
Problem sheet 2

1. Derivation of the boundary layer equations

Rewrite the two-dimensional Navier–Stokes equations in terms of the non-dimensional and scaled variables,

$$x' = \frac{x}{L}, \quad y' = Re^{1/2}\frac{y}{L}, \quad u' = \frac{u}{U}, \quad v' = Re^{1/2}\frac{v}{U}, \quad P' = \frac{P}{\rho U^2},$$

where $Re = UL/\nu$. By taking the limit $Re \to \infty$ with fixed u', $\partial u'/\partial x'$, etc., derive the boundary layer equations in their non-dimensional form.

2. Integral Momentum Equation

Starting from the boundary layer equations

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \\ 0 &= -\frac{\partial P}{\partial y}, \end{aligned}$$

show that

$$\int_0^\infty \left[u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} - u_e'\frac{du_e'}{dx'}\right] dy' = -\frac{1}{2}c_f Re^{1/2},$$

where c_f is the skin friction coefficient, the primed variables are non-dimensional and $u' \to u_e'(x')$ as $y' \to \infty$.

Hence, by using the continuity equation and eliminating v', deduce that

$$\int_0^\infty \frac{\partial}{\partial x'}[u'(u_e' - u)]dy' + \frac{du_e'}{dx'}\int_0^\infty (u_e' - u')dy' = \frac{1}{2}c_f Re^{1/2},$$

i.e.

$$\frac{1}{2}c_f Re^{1/2} = \frac{d}{dx'}(u_e'^2\theta) + \delta^* u_e'\frac{du_e'}{dx'},$$

where

$$\theta = \int_0^\infty \frac{u'}{u_e'}\left(1 - \frac{u'}{u_e'}\right) dy'$$

is the momentum thickness, and

$$\delta^* = \int_0^\infty \left(1 - \frac{u'}{u_e'}\right) dy'$$

is the displacement thickness.

P.T.O.

3. Boundary layer on curved surfaces

Compare the two-dimensional boundary layers on plane and curved surfaces below the same inviscid velocity distribution $u_0(x)$, where x is the curvilinear coordinate along the surface. Find the order of magnitude of the pressure variation across the boundary layer for the latter case, by considering the balance between the pressure gradient and the centrifugal force associated with the curved flow. Hence, show that the boundary layer equations are the same for the two cases provided that $\delta/R \ll 1$, where δ is the boundary layer thickness and R the radius of curvature of the surface.

4. Singular limit and concept of boundary layer

(This example requires techniques learnt in the perturbation methods course.)

(a) Solve the equation

$$\epsilon w'' + w' = 0, \quad \text{with} \ \ w(0) = 1, \ \ w(1) = 2, \text{for} \ \ \epsilon \ll 1.$$

Examine and write down the solutions in the regions

- $$\frac{x}{\epsilon} = X = O(1),$$
- $$x \gg \epsilon.$$

Deduce that a boundary layer of thickness $O(\epsilon)$ exists at $x = 0$.

(b) Sove the same equation but now using matched expansions and obtain the leading order inner and outer solutions.