Hydrodynamic stability theory

Problem sheet 2. Further linear stability.

Q1. Consider the two-dimensional Navier-Stokes equations for an incompressible fluid:

$$
\begin{aligned}
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right], \\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right].
\end{aligned}
\tag{1}
$$

Here $\mathbf{u} = (u, v)$ are the components of the velocity vector in the (x, y) directions, t is time, p is the pressure, and Re is a constant. Verify that $\mathbf{u} = (1 - y^2, 0)$ satfisfies the equations exactly provided that

$$p = p_B = -\frac{2x}{Re} + constant.$$

Consider small disturbances to this basic state and by writing

$$(u, v, p) = (1 - y^2, 0, p_B) + \epsilon[\tilde{u}(x, y, t), \tilde{v}(x, y, t), \tilde{p}(x, y, t)],$$

where $|\epsilon| << 1$, obtain the linearised stability equations. Assume a normal mode form and give the corresponding stability equations in this case.

Q2. The Navier-Stokes equations in cylindrical polar coordinates (r, θ, z) are given by

$$
\begin{aligned}
&\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0, \\
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{Re}\left[\triangle u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right], \\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{Re}\left[\triangle v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta}\right], \\
&\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{Re}\triangle w,
\end{aligned}
\tag{2}
$$

where

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

Here $\mathbf{u} = (u, v, w)$ are the components of the velocity vector in the (r, θ, z) directions, p is the pressure, and Re is a constant.

Show that $u = 0, v = V(r), w = 0, p = P(r)$ satisfies the equations provided

$$\frac{dP}{dr} = \frac{V^2}{r}, \quad \frac{d}{dr}\left(\frac{d}{dr} + \frac{1}{r}\right)V = 0.$$

Hence deduce the possible forms for $V(r)$. By writing disturbances to this steady state in the form

$$(u, v, w, p) = (0, V(r), 0, P(r)) + \epsilon[\tilde{u}(r, \theta, z, t), \tilde{v}(r, \theta, z, t), \tilde{w}(r, \theta, z, t), \tilde{p}(r, \theta, z, t)],$$

where $|\epsilon| << 1$, obtain the linearised stability equations.

Q3. Derive the relationship between a and l that is required for

$$f(x, y) = 2\cos(lx\sqrt{3})\cos(ly) + \cos(2ly)$$

to satisfy

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = -a^2 f$$

Write $f(x, y)$ as the *sum* of three cosine terms, and hence prove that

$$f(x + \frac{4m\pi}{a\sqrt{3}}, y + \frac{4n\pi}{a}) = f(x, y) \quad (n, m = 0, \pm 1, \pm 2, \ldots).$$

By using a polar coordinate representation of $f(x, y)$, show that there is symmetry under the transformation $\theta \to \theta + \frac{\pi}{3}$, where θ is the polar angle. What pattern would you expect to see in the fluid if this mode was realised in an experiment?

[Hint: You can use $2\cos A\cos B = \cos(A - B) + \cos(A + B)$.]