Hydrodynamic stability theory

Problem sheet 3. Bifurcation theory.

Q1. Consider a system defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x; a),$$

where a is a real parameter. Suppose that a steady solution $x = x_B$ exists, such that

$$F(x_B; a) = 0.$$

By introducing $x = x_B + \tilde{x}$ and linearizing about the steady state, show that the basic state is stable/unstable for $F'(x_B) < 0$ and $F'(x_B) > 0$ respectively.

Q2. [A Saddle-node Bifurcation] Consider a system defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a - x^2.$$

Discuss the steady states of this system, and their linear stability properties.

Consider the case $a = 0$ and solve the resulting *nonlinear* system analytically to show that for $x(t = 0) = x_0$,

$$x(t) = \frac{x_0}{1 + x_0 t}.$$

Discuss the range of possible evolutions for $x(t)$.

Q3. [A Transcritical Bifurcation] Consider a system defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - bx^2.$$

Solve the system exactly to show that

$$\frac{1}{x(t)} = \begin{cases} \frac{b}{a} + (\frac{1}{x_0} - \frac{b}{a})e^{-at} & a \neq 0 \\ \frac{1}{x_0} + bt & a = 0 \end{cases}$$

where $x_0 = x(t = 0)$.

Discuss the range of possible evolutions for $x(t)$ for different a and bx_0.