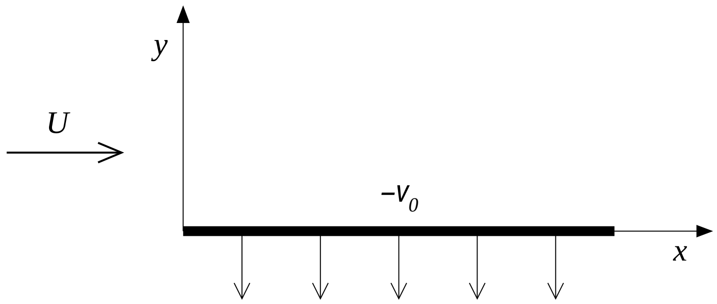
Laminar Boundary Layers

Problem sheet 4. Exact solutions.

1. Suction flow



Consider high Reynolds number flow over a porous flat plate at zero angle of incidence. A uniform suction perpendicular to the plate is accounted for by a normal, constant velocity component $-v_0$ on the surface of the plate. Far upstream of the plate, the flow has a constant free stream velocity U. The fluid has density ρ, dynamic viscosity μ, and $\nu = \mu/\rho$ is the kinematic viscosity. The equations governing the boundary layer that forms along the surface of the plate are

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{dP}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \end{aligned} \qquad (1)$$

with $dP/dx = 0$, as usual for a flat plate.

(a) State the boundary conditions appropriate for this problem. Seek a solution to (1) in which the velocity u is independent of x, *i.e.* $\partial u/\partial x = 0$ (asymptotic flow profile). Show that the solution is

$$u(y) = U\left[1 - \exp\left(-\frac{v_0}{\nu}y\right)\right].$$

(b) Calculate the displacement thickness δ^* and the momentum thickness θ. Show that the skin friction coefficient c_f is equal to

$$c_f = 2\frac{v_0}{U}.$$

(c) Show that the asymptotic flow profile calculated above is actually an exact solution of the full Navier-Stokes equations.

2. <u>Axisymmetric jet (January 2003 exam.)</u>

The dimensional boundary layer equations for the flow in a laminar, incompressible, axisymmetric jet leaving a small aperture are

$$\begin{aligned} \frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial r}(rv) &= 0, \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} &= \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \end{aligned}$$

where (x, r) denote coordinates along and perpendicular to the axis of the jet, (u, v) are the corresponding velocity components and ν is the kinematic viscosity. The stream function is defined by

$$ru = \frac{\partial \Psi}{\partial r}, \quad rv = -\frac{\partial \Psi}{\partial x}.$$

In addition,

$$\int_0^\infty u^2 r dr = M,$$

where M is a constant (independent of x), proportional to the momentum of the jet.

(a) State the boundary conditions for u and v.

(b) Looking for similarity solutions of the form

$$\Psi = \nu x^p g(\eta), \quad \eta = \frac{r}{x^q},$$

deduce that

$$p = 1, \quad q = 1,$$

and show that g satisfies the equation

$$-\frac{d}{d\eta}\left(\frac{gg'}{\eta}\right) = \frac{d}{d\eta}\left(g'' - \frac{g'}{\eta}\right).$$

(c) <u>Additional question (not part of the January 2003 exam):</u>

By integrating the equation in part (b) once, we get

$$\eta g'' - g' + gg' = 0.$$

Using this, together with $g(0) = g'(0) = 0$, show that

$$u = \frac{2\,\nu\,\alpha^2}{x\,(1 + \frac{1}{4}\,\alpha^2\eta^2)^2},$$

where

$$M = \frac{8}{3}\nu^2\,\alpha^2.$$