Hydrodynamic stability theory

Problem sheet 4. Further bifurcation theory.

Q1. Consider the dynamical system defined by

$$\frac{dx}{dt} = -y + (a - x^2 - y^2)x, \quad \frac{dy}{dt} = x + (a - x^2 - y^2)y. \tag{1}$$

(i) Set $x(t) + iy(t) = r(t)e^{i\theta(t)}$, and derive dynamical equations for $r(t), \theta(t)$.

(ii) Ignoring for now the phase angle θ, use the dynamical equation for r to plot a bifurcation diagram with r on the vertical axis and a on the horizontal.

(iii) Now reinstate the dynamics of θ, to sketch a 3D bifurcation diagram with axes (x, y) and a.

(iv) Taking 2D slices through this 3D diagram at a fixed $a < 0$, and a fixed $a > 0$, sketch the dynamical evolution in the $(x - y)$ plane for these two cases, labelling any foci or limit cycles according to whether they are stable or unstable.

Q2. Consider the dynamical system defined by

$$\frac{dx}{dt} = r\ln(x) + x - 1. \tag{2}$$

Show that $x = 1$ is a stationary solution for all r. By expanding in small $\tilde{x} = x - 1$, show that the system undergoes a transcritical bifurcation at $r_c = -1$. Setting $\tilde{x} = aX$, demonstrate a reduction to the "normal form" for a transcritical bifurcation,

$$\frac{dX}{dt} = RX - X^2, \tag{3}$$

for a particular choice of a, to be found. Give also an expression for $R(r)$.

Q3. Consider the dynamical system defined by

$$\frac{dx}{dt} = f(x; a) \tag{4}$$

in which $f(x; a) = ax - x^2$. Focusing only on values of $x \geq 0$, sketch $f(x)$ versus x for $a < 0$ and $a > 0$. Mark with circles on the x axis the locations of any stationary points x^* in the dynamics, at which $dx/dt = 0$. Give expressions for x^* in terms of a. Also indicate via arrows on the x axis the direction of the dynamical evolution (*i.e.* left if $\dot{x} < 0$ and right if $\dot{x} > 0$). Hence, mark each stationary point according to whether it is stable or unstable. What type of bifurcation does this system exhibit as a passes through zero? Plot a bifurcation diagram with $x \geq 0$ on the vertical axis and a on the horizontal.

Q4. A number of hydrodynamic systems can be shown to become chaotic through a route referred to as a "period doubling cascade". In such flows, a sequence of bifurcations is found that lead to an eventual aperiodic flow beyond a critical parameter value. A simple difference equation that behaves in the same manner is

$$x_{n+1} = \lambda x_n(1 - x_n),$$

where we consider $\lambda > 0$ and initial values, x_0, in the region $[0, 1]$.

(i) Show that this difference equation has two stationary solutions $x_n = 0$ and $x_n = 1 - \lambda^{-1}$.

(ii) Show that the trivial stationary state $\lambda_n = 0$ is stable for all $0 < \lambda < 1$, whilst the second stationary solution is stable for $\lambda \in (1, 3)$.

(iii) [Slightly harder!] Show that a period-2 solution (i.e., a periodic solution such that $x_{n+2} = x_n$) exists if $\lambda > 3$, where the two states are given by the $\pm$ solutions of

$$2x_n = 1 + \frac{1}{\lambda} \pm \frac{1}{\lambda}\sqrt{\lambda^2 - 2\lambda - 3}.$$

(iv) For a general difference equation of the form

$$x_{n+1} = F(x_n),$$

suppose that a particular period-2 solution exists such that $X_{n+1} = F(X_n)$ and $X_{n+2} = X_n$. Show that such a solution is stable provided that

$$|F'(X_n)F'(X_{n+1})| < 1.$$

[For information/interest only] Using this condition, it can be shown that the period-2 solution found in (iii) above becomes unstable for $\lambda > \lambda_2 = 1 + \sqrt{6}$, with a period-4 solution arising here. Subsequently, this period-4 solution loses stability for $\lambda > \lambda_3$, and a period-8 solution arises *etc.* There exists a limiting point $\lambda_\infty \approx 3.57$, furthermore the ratio of $|\lambda_n - \lambda_{n-1}|/|\lambda_{n+1} - \lambda_n| \to 4.699..$ as $n \to \infty$. This constant is called the Feigenbaum constant and is universal, having been verified in many systems (not just this equation!).