Hydrodynamic stability theory

Problem sheet 5. The Stuart–Landau and Ginzburg–Landau equations.

Q1. In the lecture notes we derived the Stuart–Landau equation governing the weakly nonlinear evolution of a perturbation to the basic state $\Phi_B = \eta$ of the Eckhaus equation. The complex amplitude $A(\tau)$ was shown to satisfy

$$A_\tau = \frac{R_1}{R_{cm}^2}(\pi^2 + k_{cm}^2)A + \beta A|A|^2,$$

when the control parameter R is perturbed about the minimum critical value, such that

$$R = R_{cm} + \delta R_1,$$

and a solution is sought in the form

$$\Phi = \eta + \delta^{1/2}\phi_1 + \delta\phi_2 + \delta^{3/2}\phi_3 + \ldots.$$

(1) Determine explicitly the real constant β appearing in the Stuart–Landau equation above in terms of the critical wavenumber and Reynolds number (k_{cm}, R_{cm}).

(2) Consider the more general case of $R = R_c + \delta R_1$: *i.e.*, an expansion around a general neutral point for the $n = 1$ disturbance, rather than the minimum value R_{cm}. Is the same expansion procedure valid? What would be the significance of a change in sign of β?

Q2. Consider a flow in a region $y \in [0, \pi]$, governed by the following equation:

$$\frac{\partial U}{\partial t} = f(U) + \frac{1}{R}\frac{\partial^2 U}{\partial y^2}$$

where $R > 0$ is a real parameter and $U = 0$ at $y = 0, \pi$.

(i) For $f(U) = U + U^3$, write down the linearized stability equation for a perturbation to the trivial basic state $U_{\mathrm{B}} = 0$. Hence determine the values of R at which bifurcation points can be found. If the lowest value of R for which a bifurcation occurs is denoted by $R = R_c$, find R_c.

(ii) For a perturbation about the minimum critical value $R = R_c + \epsilon R_1$, $(0 < \epsilon \ll 1)$ and an expansion in the form

$$U = U_{\mathrm{B}} + \epsilon^{1/2}A(\tau)\sin(y) + \epsilon U_2(y,\tau) + \epsilon^{3/2}U_3(y,\tau) + \cdots,$$

where $\tau = \epsilon t$, derive an amplitude equation for the perturbation amplitude $A(\tau)$. Classify the bifurcation type and draw a bifurcation diagram.

Q3. Consider a flow governed by the nonlinear equation

$$U_t - U^2 = U_{xx},$$

where U satisfies the boundary conditions $U_x = 0$ on $x = 0, 1$, and initial condition $U(x, t = 0) = U_0(x)$.

Introduce the Fourier cosine series

$$U = \frac{1}{2} \sum_{n=-\infty}^{n=+\infty} a_n(t) e^{in\pi x},$$

where $a_{-n} = a_n$ are real functions of the time coordinate t.

Show that

$$\frac{\mathrm{d}a_n}{\mathrm{d}t} = -n^2\pi^2 a_n + \frac{1}{2} \sum_{m=-\infty}^{m=+\infty} a_m a_{n-m},$$

and hence that

$$\frac{\mathrm{d}a_0}{\mathrm{d}t} \geq \frac{1}{2} a_0^2.$$

Show that, if $a_0(t = 0) > 0$ then

$$a_0(t) \geq \frac{2}{t_0 - t},$$

where t_0 is a positive constant. Hence conclude that if $\int_0^1 U_0(x)\mathrm{d}x > 0$ the flow approaches a singularity as $t \to t_0$.