3 The boundary layer equations

Having introduced the concept of the boundary layer (BL), we now turn to the task of deriving the equations that govern the flow inside it. We focus throughout on the case of a 2D, incompressible, steady state of constant viscosity. We start for simplicity by considering a flat plate of length L with normal in the y direction, subject to a uniform free stream far from the plate of velocity U in the x direction. We return below to discuss the extension of our results to curved surfaces.

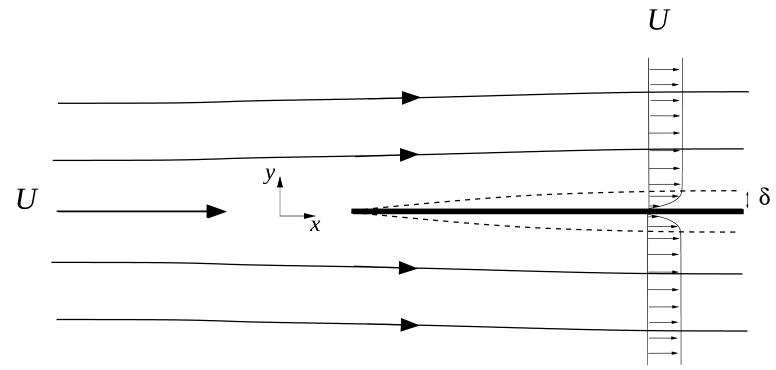

Figure 5: The boundary layer along a flat plate at zero incidence. (After Schlichting, 'Boundary Layer Theory', McGraw Hill.)

3.1 Derivation

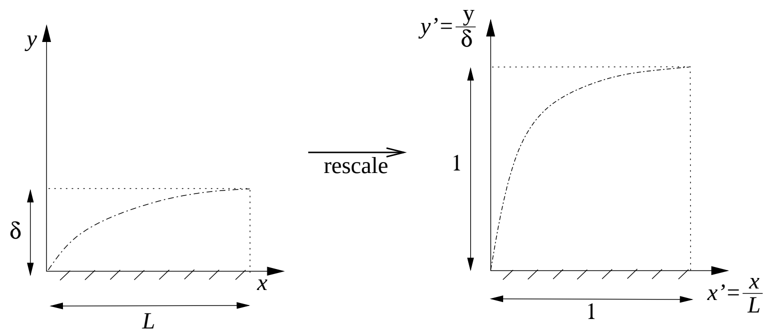

Figure 6: Boundary layer transformation, to render the flow variables $O(1)$. Dot–dashed line shows the thickness of the boundary layer.

Looking back at the derivation of the non-dimensional N–S Eqns. 26 to 28, we recall that we expressed all lengths and velocities in the natural units L and U. The key point in boundary layer theory, however, is the existence of two different length scales:

- the characteristic size of the object L,
- the typical thickness of the boundary layer $\delta \ll L$. Recall Fig. 4.

(The smallness of δ will emerge from the analysis that follows.) Accordingly, we now take L as the natural unit of length in the x direction, and δ as that in the y direction, defining:

$$x' = \frac{x}{L}, \;\; y' = \frac{y}{\delta}. \tag{36}$$

The actual thickness of the BL will actually vary with distance along the plate (Figs. 5 and 6; not shown in Fig. 4), so we now choose δ to be the value at the trailing edge, $x = L$, Fig. 6. Thus, both L and δ are constants in what follows. Similarly, we expect different scales for the x and y components of velocity, and choose:

$$u' = \frac{u}{U}, \;\; v' = \frac{v}{V}, \tag{37}$$

where U and V are likewise constants. δ and V are as yet unknown: they will be determined by the following analysis. Finally we adimensionalise the pressure in the usual way:

$$P' = \frac{P}{\rho U^2}. \tag{38}$$

The basic strategy in such a transformation is to render all flow variables x', y', u', v', P' and all derivatives ($\partial u'/\partial x'$, *etc.*) of order unity, Fig. 6. By expressing the flow equations in terms of these rescaled variables, any terms that are negligible will then reveal themselves by having a small prefactor. (This was precisely the procedure adopted to identify regimes of high and low Re in Sec. 2.1.) We this in mind, we now apply the transformation 36 to 38 to each of the flow Eqns. 22 to 24 in turn.

Continuity:

In terms of the above variables the continuity equation, Eqn. 22, becomes

$$\frac{U}{L}\frac{\partial u'}{\partial x'} + \frac{V}{\delta}\frac{\partial v'}{\partial y'} = 0. \tag{39}$$

For $\partial u'/\partial x'$ and $\partial v'/\partial y'$ to both be of order unity, we require

$$\frac{L}{U}\frac{V}{\delta} = O(1), \;\; \text{and so choose} \;\; V = U\frac{\delta}{L}. \tag{40}$$

The non-dimensional continuity equation is then

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0. \tag{41}$$

Anticipating the result (derived below) that the BL thickness δ is small, we see from Eqn. 40 that the characteristic vertical velocity V is also small, as we would expect intuitively.

x-momentum

The horizontal component of the force balance equation, Eqn. 23, becomes

$$\rho U\frac{1}{L}Uu'\frac{\partial u'}{\partial x'} + \rho\frac{U\delta}{L}\frac{1}{\delta}Uv'\frac{\partial u'}{\partial y'} = -\rho\frac{U^2}{L}\frac{\partial P'}{\partial x'} + \mu\left(U\frac{1}{L^2}\frac{\partial^2 u'}{\partial x'^2} + U\frac{1}{\delta^2}\frac{\partial^2 u'}{\partial y'^2}\right). \tag{42}$$

On dividing through by $\rho U^2/L$, we obtain the non-dimensional form

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\frac{\partial P'}{\partial x'} + \frac{1}{Re}\frac{\partial^2 u'}{\partial x'^2} + \frac{(L/\delta)^2}{Re}\frac{\partial^2 u'}{\partial y'^2}, \tag{43}$$

in which the Reynolds number $Re = UL\rho/\mu$, as usual. In the limit $Re \to \infty$, the coefficient of $\partial^2 u'/\partial x'^2$ becomes small, so this term can be ignored. In order for the remaining viscous term to be retained in accordance with the discussion of Sec. 2, the coefficient of $\partial^2 u'/\partial y'^2$ must remain of order unity. Therefore we require

$$\frac{\delta}{L} = \mathrm{O}(Re^{-1/2}), \text{ and so choose } \frac{\delta}{L} = Re^{-1/2}. \tag{44}$$

The $x-$momentum equation then becomes

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\frac{\partial P'}{\partial x'} + \frac{\partial^2 u'}{\partial y'^2}. \tag{45}$$

Eqn. 44 is an important result: it tells us that the BL becomes infinitesimally thin in the limit $Re \to \infty$, as anticipated above. Euler's equations of inviscid flow, valid outside the BL, therefore apply right down to (an infinitesimal distance above) the body surface. This will allow us below to find an expression for the term $\partial P'/\partial x'$ on the RHS of Eqn. 45.

y-momentum:

The vertical component of the force balance equation, Eqn. 24, becomes

$$\rho U\frac{1}{L}\frac{U\delta}{L}u'\frac{\partial v'}{\partial x'} + \rho\frac{U\delta}{L}\frac{1}{\delta}\frac{U\delta}{L}v'\frac{\partial v'}{\partial y'} = -\rho\frac{U^2}{\delta}\frac{\partial P'}{\partial y'} + \mu\left(\frac{U\delta}{L}\frac{1}{L^2}\frac{\partial^2 v'}{\partial x'^2} + \frac{U\delta}{L}\frac{1}{\delta^2}\frac{\partial^2 v'}{\partial y'^2}\right). \tag{46}$$

On division throughout by $\rho U^2/\delta$, we obtain

$$\left(\frac{\delta}{L}\right)^2\left\{u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'}\right\} = -\frac{\partial P'}{\partial y'} + \frac{1}{Re}\left\{\left(\frac{\delta}{L}\right)^2\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right\}. \tag{47}$$

Neglecting small terms, $O(Re^{-1})$, we get simply

$$0 = -\frac{\partial P'}{\partial y'}. \tag{48}$$

This tells us that the pressure does not vary vertically through the BL: $P' = P'(x')$ only. The BL therefore experiences, through its entire depth, the pressure field that is imposed at its exterior edge, $P'(x') = P'_e(x')$. On the exterior, the flow is governed by Euler's inviscid equations. As noted above, we solve these in a domain that continues essentially right down to the solid surface, since the BL is infinitesimally thin. The BL however plays the crucial role of rendering Euler's equations free of the no-slip condition, Fig. 4 (leaving only no-permeation). For our purposes, we assume that this inviscid calculation has already been performed, giving $u' = u'_e(x'), v' = 0$ on the exterior edge of the BL: *i.e.*, the function u_e prescribing the inviscid slipping velocity is given to us. $P'(x') = P'_e(x')$ can be expressed in terms of u_e via the x component of the inviscid momentum equation, Eqn. 34

$$u'_e\frac{du'_e}{dx'} = -\frac{dP'_e}{dx'}. \tag{49}$$

Substituting this into Eqn. 45, and recalling Eqn. 41, we get finally the **adimensional boundary layer equations**:

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0, \tag{50}$$

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = u'_e\frac{du'_e}{dx'} + \frac{\partial^2 u'}{\partial y'^2}, \tag{51}$$

with the **transformation variables**

$$x' = \frac{x}{L},\ y' = Re^{1/2}\frac{y}{L},\ u' = \frac{u}{U},\ v' = Re^{1/2}\frac{v}{U}\ \ Re = \frac{UL}{\nu} \tag{52}$$

and **boundary conditions**:

- On the body surface, $y' = 0$: no slip and no permeation, $u' = v' = 0$.
- At the exterior edge of the boundary layer[3], $y' \to \infty$, the velocity must match the surface slipping velocity calculated according to inviscid theory: $u' \to u'_e(x')$.

Because the boundary layer equations are independent of Re, the only information required to solve them is $u'_e(x')$, which depends on the shape of the body and its orientation relative to the free stream[4].

3.2 The stream function

The equations just derived can be simplified by noticing that the continuity equation is automatically satisfied (in 2D) if we define a stream function Ψ such that

$$u = \frac{\partial \Psi}{\partial y},\ v = -\frac{\partial \Psi}{\partial x}. \tag{53}$$

Plug these into Eqn. 22 to verify this. In dimensionless form

$$u' = \frac{\partial \Psi'}{\partial y'},\ v' = -\frac{\partial \Psi'}{\partial x'}, \tag{54}$$

in which Ψ' is defined by

$$\Psi' = \Psi\frac{Re^{1/2}}{UL} = \frac{\Psi}{\sqrt{\nu UL}}. \tag{55}$$

(Check this scaling of Ψ by plugging the usual transformations $x = x'L$, $y = y'L/Re^{1/2}$, $u = u'U$, $v = v'U/Re^{1/2}$ together with $\Psi = \Psi'/\alpha$ into Eqn. 53 to see that we get Eqn. 54 provided α matches the prefactor in Eqn. 55.)

Having (automatically) solved the continuity equation 50, we now just need to solve the momentum equation 51. Expressed in terms of the stream function, this is:

$$\frac{\partial \Psi'}{\partial y'}\frac{\partial^2 \Psi'}{\partial x' \partial y'} - \frac{\partial \Psi'}{\partial x'}\frac{\partial^2 \Psi'}{\partial y'^2} = u'_e\frac{du'_e}{dx'} + \frac{\partial^3 \Psi'}{\partial y'^3}. \tag{56}$$

The boundary conditions are, as usual:

[3]The limit $y' \to \infty$, which takes us to the inviscid side of the infinitesimally thin boundary layer, is very different from the limit $y \to \infty$, which truly takes us far from the body, out to the free stream.

[4]Strictly, the equations just derived become exact only in the limit $Re \to \infty$. For large Re, they represent a first order approximation. The scaling used to obtain them implies an $O(Re^{-1/2})$ correction in the inviscid region, for example.

- No slip and no permeation at the solid surface,

$$\frac{\partial \Psi'}{\partial y'} = \frac{\partial \Psi'}{\partial x'} = 0 \text{ at } y' = 0. \quad (57)$$

- Inviscid slipping solution on the exterior edge of the boundary layer,

$$u' = \frac{\partial \Psi'}{\partial y'} \to u'_e(x') \text{ as } y' \to \infty. \quad (58)$$

We will employ the stream function extensively throughout the course.

3.3 Curved surfaces

The boundary layer equations just obtained appear rather restrictive, because they were derived for a flat surface. What about curved surfaces? Provided δ remains small compared with the local radius of curvature of the surface, it is possible to show that Eqns. 50 and 51 still apply. The only modification is geometrical: x becomes the coordinate measured along the curved surface and y the normal to the local surface direction. See figure 7 (which also shows the way in which the boundary layer can continue as a wake behind the obstacle).

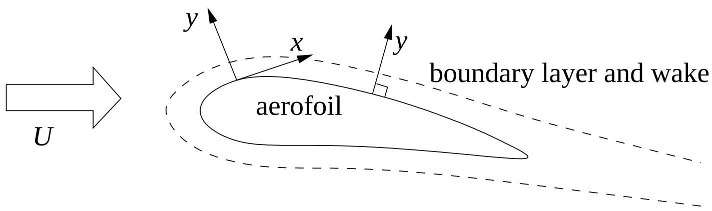

Figure 7: Boundary layer around a curved surface (and the wake behind it).

3.4 Forces on the body

We now consider the forces exerted by the flow on the body. These are calculated by integrating the **normal** and **tangential** stress components over the surface to find the **lift** and **drag** forces respectively.

We recall that the dimensional stress tensor, Eqn. 17 can be written as

$$\Pi_{ij} = -P\delta_{ij} + p_0\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \quad (59)$$

When integrated over the surface, the static pressure p_0 yields a buoyancy force $\rho g V_b$ (where V_b is the volume of the body) that is generally insignificant, at high Re, compared with the other forces of lift and drag. In any case, it is independent of the flow, and we neglect it. The (remaining) stress tensor may be non-dimensionalised using the boundary layer transformations 52 to give

$$\frac{\Pi_{xx}}{\rho U^2} = -P' + \frac{2}{Re}\frac{\partial u'}{\partial x'}, \qquad \frac{\Pi_{yy}}{\rho U^2} = -P' + \frac{2}{Re}\frac{\partial v'}{\partial y'}, \quad (60)$$

$$\frac{\Pi_{xy}}{\rho U^2} = \frac{\Pi_{yx}}{\rho U^2} = \frac{1}{Re^{1/2}} \left(\frac{\partial u'}{\partial y'} + \frac{1}{Re} \frac{\partial v'}{\partial x'} \right). \tag{61}$$

For large Re, these reduce to

$$\frac{\Pi_{xx}}{\rho U^2} = \frac{\Pi_{yy}}{\rho U^2} = -P', \quad \frac{\Pi_{xy}}{\rho U^2} Re^{1/2} = \frac{\partial u'}{\partial y'}. \tag{62}$$

The normal stresses are thereby seen to reduce to the pressure P, and the viscous shear stress Π_{xy} to $\mu \partial u / \partial y$. The **skin friction coefficient** at the body surface $y = 0$ is defined as a measure of the shear stress relative to the natural pressure scale ρU^2:

$$c_f = \frac{(\Pi_{xy})_{y=0}}{\frac{1}{2}\rho U^2} = \frac{\mu (\partial u / \partial y)_{y=0}}{\frac{1}{2}\rho U^2}. \tag{63}$$

(The factor half is for convention.) Using transformations 52 we see

$$c_f Re^{1/2} = 2 \left(\frac{\partial u'}{\partial y'} \right)_{y'=0} = f(x') \text{ alone, for geometrically similar bodies.} \tag{64}$$