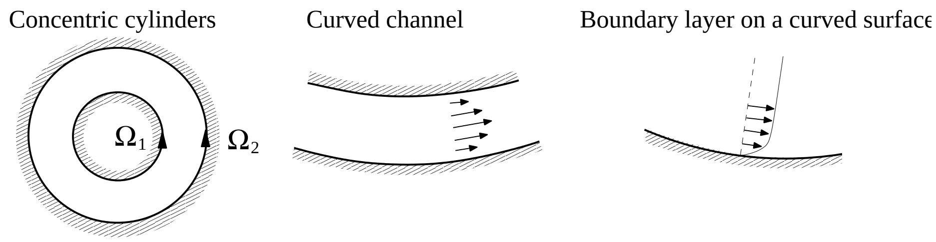
3 Centrifugal instabilities

Flows with curved streamlines, such as those sketched below, can be unstable due to the centrifugal effects of rotation. Here we focus on centrifugal instabilities in **inviscid fluids**. Our main focus is Rayleigh's criterion for the instability of a basic swirling flow with an arbitrary dependence of angular velocity $\Omega(r)$ on the distance r from the axis of rotation. This states that

$$\Phi(r) < 0 \quad \text{for instability, where} \quad \Phi = \frac{1}{r^3}\frac{d}{dr}\left(r^4\Omega^2\right). \tag{88}$$

In Sec. 3.1 we motivate (88) using a simple physical argument. In Sec. 3.2, we prove it via linear stability analysis. In Sec. 3.3, we apply it to flow between concentric cylinders. Note an analogy between these curvature driven instabilities and the thermal instabilities discussed above. Fluid elements are forced outward by centrifugal effects in one case; and upwards by their buoyancy in the other.



The **governing equations** are as follows. For inviscid fluids, and in the absence of body forces, the Navier-Stokes equations (1, 2) reduce to Euler's equations:

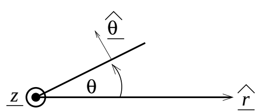
Continuity

$$\nabla \cdot \mathbf{u} = 0. \tag{89}$$

Momentum balance

$$\rho\left[\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = -\nabla p. \tag{90}$$

Throughout this section, we use cylindrical coordinates



and consider **axisymmetric flows**, which can depend on r and z but not θ. Componentwise we then have

Continuity

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0 \tag{91}$$

Momentum balance

$$\frac{Du_r}{Dt} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{92}$$

$$\frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r} = 0 \tag{93}$$

$$\frac{Du_z}{Dt} = -\frac{1}{\rho}\frac{\partial z}{\partial z} \tag{94}$$

in which the material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + u_z\frac{\partial}{\partial z}. \tag{95}$$

3.1 Rayleigh's criterion for inviscid instability

Consider an initially laminar azimuthal flow,

$$\mathbf{u} = u_\theta(r)\hat{\theta}, \tag{96}$$

with an arbitrary dependence of azimuthal velocity $u_\theta = r\Omega(r)$ on r. Rayleigh provided a criterion to distinguish between stable and unstable distributions of the angular velocity $\Omega(r)$ using a simple physical argument, which we now describe.

Noting that the $\hat{\theta}$ component of momentum balance, Eqn. 93, can be rewritten as

$$\frac{D}{Dt}(ru_\theta) = 0 \tag{97}$$

we see that the quantity $H \equiv ru_\theta$, which is the angular momentum, is conserved for each material element. This is to be expected in the absence of viscous dissipation.

Associated with the azimuthal motion is a kinetic energy per unit volume of

$$\frac{1}{2}\rho u_\theta^2 = \frac{1}{2}\frac{\rho H^2}{r^2}. \tag{98}$$

Now consider two volume elements of equal volumes dV at radial locations $r = r_1$ and $r = r_2$ with $r_2 > r_1$. Their combined kinetic energy is

$$E = \frac{1}{2}\rho\left(\frac{H_1^2}{r_1^2} + \frac{H_2^2}{r_2^2}\right)dV. \tag{99}$$

Now imagine that these elements swap positions. By virtue of Eqn. 97, each keeps its own angular momentum. After the swap, their combined energy is thus

$$E_{\text{new}} = \frac{1}{2}\rho\left(\frac{H_2^2}{r_1^2} + \frac{H_1^2}{r_2^2}\right)dV. \tag{100}$$

So the swap has resulted in an energy change

$$\Delta E \propto \left(H_2^2 - H_1^2\right)\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right). \tag{101}$$

If the swap has released energy ($\Delta E < 0$, $H_1^2 > H_2^2$), the laminar base flow will be unstable to such swaps. Thus, the criterion for instability is that H^2 decreases with r:

$$\frac{d}{dr}H^2 < 0 \quad \text{for instability.} \tag{102}$$

Recalling that $H = ru_\theta = r^2\Omega$, the condition for instability is finally seen to be

$$\frac{d}{dr}\left(r^4\Omega^2\right) < 0 \quad \text{for instability.} \tag{103}$$

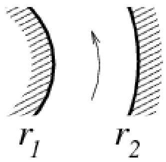
This is consistent with our original statement (88) above.

3.2 Proof via a linear stability analysis

Having motivated Rayleigh's condition (88) via a simple physical argument, we now prove it by means of a linear stability analysis.

3.2.1 Governing equations and boundary conditions

The governing equations of inviscid axisymmetric flow have already been specified: (91) to (95) above. We now apply these to flow between two impermeable boundaries located at $r = r_1, r_2$.



3.2.2 Base state

As above, for the base state we consider a laminar swirling azimuthal flow

$$\mathbf{u}_\mathrm{B} = \begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix} = \begin{pmatrix} 0 \\ r\Omega(r) \\ 0 \end{pmatrix} \tag{104}$$

with an arbitrary dependence of the angular velocity $\Omega(r)$ on radius, r.

3.2.3 Small perturbation

We now subject the base state to a small perturbation, still assuming axisymmetry ($\partial_\theta \cdots = 0$)

$$\mathbf{u} = \begin{pmatrix} 0 \\ r\Omega(r) \\ 0 \end{pmatrix} + \delta \begin{pmatrix} \tilde{u}(r,z,t) \\ \tilde{v}(r,z,t) \\ \tilde{w}(r,z,t) \end{pmatrix} \tag{105}$$

with an analogous expression for the pressure p. As usual, the parameter δ is small.

3.2.4 Linearise the equations

Following our usual procedure, we substitute (105) into the governing equations (91) to (95) and expand in powers of δ. Neglecting terms $O(\delta^2)$ and higher, we get the linearised equations

Continuity

$$\left(\partial_r + \frac{1}{r}\right)\tilde{u} + \partial_z \tilde{w} = 0. \tag{106}$$

Momentum balance

$$\partial_t \tilde{u} - 2\Omega(r)\tilde{v} = -\frac{1}{\rho}\partial_r \tilde{p} \tag{107}$$

$$\partial_t \tilde{v} + \tilde{u}\partial_r\left(\Omega(r)r\right) + \Omega(r)\tilde{u} = 0 \tag{108}$$

$$\partial_t \tilde{w} = -\frac{1}{\rho}\partial_z \tilde{p}. \tag{109}$$

3.2.5 Solve the linearised equations using normal modes

We now express the perturbation as a sum of normal modes:

$$\begin{pmatrix} \tilde{u}(r,z,t) \\ \tilde{v}(r,z,t) \\ \tilde{w}(r,z,t) \end{pmatrix} = \sum \begin{pmatrix} \hat{u}(r) \\ \hat{v}(r) \\ \hat{w}(r) \end{pmatrix} \exp\left(ikz + st\right), \tag{110}$$

with an analogous expression for the pressure. k is the wavevector in the axial direction. The linearised equations (106) to (109) then become

Continuity

$$\left(\frac{d}{dr} + \frac{1}{r}\right)\hat{u} + ik\hat{w} \quad = \quad 0. \tag{111}$$

Momentum balance

$$s\hat{u} - 2\Omega(r)\hat{v} \quad = \quad -\frac{1}{\rho}\frac{d\hat{p}}{dr} \tag{112}$$

$$s\hat{v} + \hat{u}\frac{d}{dr}\left(\Omega(r)r\right) + \Omega(r)\hat{u} \quad = \quad 0 \tag{113}$$

$$s\hat{w} \quad = \quad -\frac{ik}{\rho}\hat{p} \tag{114}$$

Our strategy now is to progressively eliminate $\hat{p}, \hat{v}$ and $\hat{w}$, leaving a single equation for $\hat{u}$. We will then use this to distinguish between stable ($s_r < 0$) and unstable ($s_r > 0$) perturbations.

First we eliminate $\hat{p}$ by taking ik(112)-$\frac{d}{dr}$(114) to get

$$ik(s\hat{u} - 2\Omega\hat{v}) - s\frac{d\hat{w}}{dr} = 0. \tag{115}$$

This leaves (111, 113, 115) in $\hat{u}, \hat{v}, \hat{w}$. From Eqn. 113 we have

$$\hat{v} = -\frac{1}{s}\left(2\Omega + r\frac{d\Omega}{dr}\right)\hat{u}, \tag{116}$$

which can be substituted into (115) to give

$$iks\hat{u} + \frac{2ik\Omega}{s}\left(2\Omega + r\frac{d\Omega}{dr}\right)\hat{u} = s\frac{d\hat{w}}{dr}. \tag{117}$$

We have now eliminated $\hat{v}$, leaving (111,117) for $\hat{u}, \hat{w}$. Multiplying (117) by ik/s, we get

$$-k^2\hat{u} - \frac{2k^2\Omega}{s^2}\left(2\Omega + r\frac{d\Omega}{dr}\right)\hat{u} = ik\frac{d\hat{w}}{dr}. \tag{118}$$

This can finally be combined with $\frac{d}{dr}$(111) to eliminate $\hat{w}$, leaving a single equation in $\hat{u}$

$$\frac{d}{dr}\left(\frac{d}{dr} + \frac{1}{r}\right)\hat{u} - k^2\hat{u} - \frac{2k^2\Omega}{s^2}\left(2\Omega + r\frac{d\Omega}{dr}\right)\hat{u} = 0. \tag{119}$$

Defining $\Phi(r)$ as in (88), this can easily be written in the simpler form

$$\frac{1}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\hat{u}\right)\right) - k^2\hat{u} = \frac{k^2}{s^2}\Phi(r)\hat{u}. \tag{120}$$

Check this as an exercise. Now multiplying across by $r\hat{u}^{(c)}$, where (c) denotes complex conjugate, and integrating from r_1 to r_2, we get

$$\int_{r_1}^{r_2} r\hat{u}^{(c)}\frac{1}{dr}\left(\frac{1}{r}\frac{d}{dr}(r\hat{u})\right)dr - k^2\int_{r_1}^{r_2} r|\hat{u}|^2 dr = \frac{k^2}{s^2}\int_{r_1}^{r_2}\Phi(r)r|\hat{u}|^2 dr. \tag{121}$$

Integrating the first term by parts we get

$$\left[r\hat{u}^{(c)}\frac{1}{r}\frac{d}{dr}(r\hat{u})\right]_{r_1}^{r_2} - \int_{r_1}^{r_2}\frac{1}{r}\left|\frac{d}{dr}(r\hat{u})\right|^2 dr - k^2\int_{r_1}^{r_2} r|\hat{u}|^2 dr = \frac{k^2}{s^2}\int_{r_1}^{r_2}\Phi(r)r|\hat{u}|^2 dr. \tag{122}$$

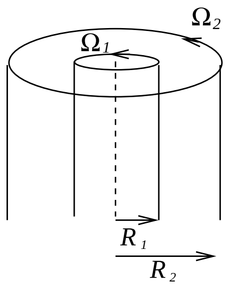
The first term is zero, because the boundaries at r_1, r_2 are impermeable. Hence, (122) has the form

$$-I_1 - k^2 I_2 = \frac{k^2}{s^2}\int_{r_1}^{r_2}\Phi(r)r|\hat{u}|^2 dr, \tag{123}$$

in which $I_1 > 0$, $I_2 > 0$. The characteristic values k^2/s^2 are therefore all negative if $\Phi > 0$ throughout the interval $r_1 < r < r_2$. In contrast, if $\Phi < 0$ in some region then we can have $s^2 > 0$ and $s_{\mathrm{r}} > 0$, denoting linear instability. This is in accordance Rayleigh's criterion (88), motivated in the previous section via a simple physical argument. If Φ is positive everywhere, however, note that we still cannot conclude stability without considering non-axisymmetric distribances as well. We do not pursue that issue further.

3.3 Taylor vortices

We now apply Rayleigh's criterion (88) to Couette flow between infinitely long concentric cylinders.



In particular, we are interested whether a basic swirl solution

$$\mathbf{u}_{\mathrm{B}} = \begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix} = \begin{pmatrix} 0 \\ v(r) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ r\Omega(r) \\ 0 \end{pmatrix} \tag{124}$$

is stable with respect to axisymmetric perturbations. The main task here is to derive the basic flow $v(r) = r\Omega(r)$. This can then be plugged directly into Rayleigh's criterion to determine stability/instability.

It can be shown that a possible[1] basic solution has the form

$$v(r) = Ar + \frac{B}{r} \tag{125}$$

[1] This is the only possible form if viscosity is present. It is the one that we shall use, even though we are considering the inviscid case in which the class of permissible solutions is actually rather wide.

in which A and B are constants. To determine these, we apply the boundary conditions:

$$\text{at} \quad r = R_1, \quad \Omega = \Omega_1 = A + \frac{B}{R_1^2} \tag{126}$$

$$\text{at} \quad r = R_2, \quad \Omega = \Omega_2 = A + \frac{B}{R_2^2}. \tag{127}$$

Solving these gives

$$B = \frac{(\Omega_1 - \Omega_2)}{\left(\frac{1}{R_1^2} - \frac{1}{R_2^2}\right)} = \Omega_1 R_1^2 \frac{\left(1 - \frac{\Omega_2}{\Omega_1}\right)}{\left(1 - \frac{R_1^2}{R_2^2}\right)}. \tag{128}$$

Now let

$$\mu = \Omega_2/\Omega_1, \quad \eta = R_1/R_2 < 1. \tag{129}$$

(The symbol η used here is obviously not related to the η used for viscosity in previous sections.) This gives

$$B = \Omega_1 R_1^2 \frac{(1-\mu)}{(1-\eta^2)} \quad \text{and} \quad A = -\Omega_1 \left(\frac{\eta^2 - \mu}{1 - \eta^2}\right). \tag{130}$$

So (125) and (130) together give the laminar base flow.

We now examine the linear stability of this base flow using Rayleigh's criterion (88). First we need to calculate

$$\begin{aligned}
\Phi(r) &= \frac{1}{r^3}\frac{d}{dr}\left(r^4\Omega^2\right) \\
&= \frac{1}{r^3}\frac{d}{dr}\left(r^4\left[A^2 + \frac{2AB}{r^2} + \frac{B^2}{r^4}\right]\right) \\
&= 4A^2\left(1 + \frac{B}{Ar^2}\right).
\end{aligned} \tag{131}$$

(Do the algebra of the last step as an exercise.) Recall that the condition for instability is $\Phi < 0$. Now $4A^2 > 0$ always, so the condition for instability is just that the quantity in brackets in (131) is negative. Expanding this using the expressions (130), we get

$$\Phi(r) = 4A^2\left(1 - \frac{(1-\mu)R_1^2}{(\eta^2 - \mu)r^2}\right). \tag{132}$$

(Again, do the algebra as an exercise.) We consider values of $\mu > 0$, corresponding to both cylinders rotating in the same sense (Ω_1 and Ω_2 having the same sign). In this case, it can be shown that

$$\Phi > 0 \quad \text{(giving stability) if} \quad \mu > \eta^2 \tag{133}$$

and

$$\Phi < 0 \quad \text{(giving instability) if} \quad \mu < \eta^2. \tag{134}$$

This is shown by the solid black line in the figure overleaf. The dashed line shows the stabilising effect of a non-zero viscosity, though we do not calculate that result here. As can be seen, the flow is linearly stable if only the outer cylinder rotates ($\Omega_1 = 0$).

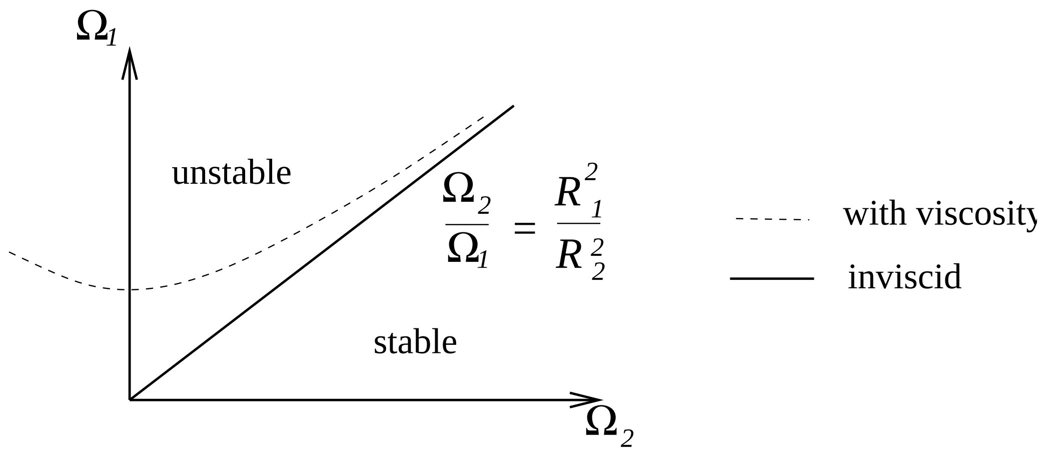


Figure 2: Linear stability/instability of axisymmetric Couette flow for different (co)rotation rates of the outer and inner cylinders.