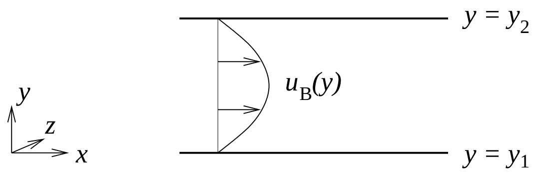
4 Shear flow instabilities

Here we consider the linear stability of a uni-directional base flow in a channel

$$\mathbf{u}_\mathrm{B} = \begin{pmatrix} u_\mathrm{B}(y) \\ 0 \\ 0 \end{pmatrix} \quad \text{in the region} \quad y \in [\,y_1, y_2\,]. \tag{135}$$



In Sec. 4.1 we derive an equation governing the linear stability of such flows with respect to 3D perturbations, for viscous fluids. This is called the Orr-Somerfeld equation. In Sec. 4.2 we discuss Squire's theorem, which tells us that we only need to consider 2D (x, y) perturbations in order to determine the *first occurrence* of instability as the Reynolds number is increased. In Sec. 4.3 we turn to inviscid fluids, and derive Rayleigh's inflexion point theorem for inviscid instability. Secs. 4.1 to 4.3 consider a general base state $u_\mathrm{B}(y)$. Finally in Sec. 4.4 we specialise $u_\mathrm{B}(y)$ to the concrete case of plane Poiseuille flow.

4.1 The Orr-Somerfeld equation

Here we derive the Orr-Somerfeld equation, which governs the linear stability of uni-directional shear flows with respect to 3D perturbations, for viscous fluids.

4.1.1 Governing equations and boundary conditions

In non-dimensional form, the Navier-Stokes equations for the incompressible flow of a viscous fluid are **mass continuity**

$$\nabla \cdot \mathbf{u} = 0, \tag{136}$$

and the **momentum equations**

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u}, \tag{137}$$

in which Re is the Reynolds number. As usual we will represent $\mathbf{u} = (u, v, w)^T$ in Cartesian components $(x, y, z)^T$. For boundary conditions, we assume no-slip and no-permeation at each wall $y = y_1, y_2$. In the limit $Re \to \infty$ of inviscid flow, the term in $\nabla^2 \mathbf{u}$ drops out of (137). We discuss this case in Sec. 4.3 below.

4.1.2 Base state

All results in Secs. 4.1 to 4.3 will be derived for a general base state $u_\mathrm{B}(y)$. In Sec. 4.4 we will consider the concrete example of planar Poiseuille flow $u_\mathrm{B}(y) = 1 - y^2$, which corresponds to flow between two infinite parallel planes at $y_1 = -1, y_2 = +1$, driven by a pressure gradient $\partial_x p_\mathrm{B} = -2/Re$.

4.1.3 Small perturbation

As usual, we subject the base state to a small perturbation, as follows:

$$\mathbf{u} = \mathbf{u}_\mathrm{B} + \delta \begin{pmatrix} \tilde{u}(y) \\ \tilde{v}(y) \\ \tilde{w}(y) \end{pmatrix} e^{i(\alpha x+\beta z-\omega t)}, \tag{138}$$

$$p = p_\mathrm{B} + \delta \tilde{p}(y) e^{i(\alpha x+\beta z-\omega t)}, \tag{139}$$

(real part understood). As usual δ is a small parameter, $|\delta| \ll 1$. The perturbation has the form of a travelling wave with wavenumber components (α, β) in the (x, z) plane; and frequency/growth rate ω. So note that here we have directly written the perturbation in normal mode form, even before linearising the equations. In previous sections, we instead linearised the equations for a general perturbation $\mathbf{u} = \mathbf{u}_\mathrm{B} + \delta\mathbf{u}(x, y, z, t)$, turning to normal modes only later to solve these linearised equations. The present approach is merely a shortcut through the same procedure.

In principle, two different kinds of instability can be discussed:

- Temporal instability – Here we assume that (α, β) are real and write $\omega = \alpha c$ where $c = c_\mathrm{r} + ic_\mathrm{i}$ is complex. This gives

$$e^{-i\omega t} = e^{-i\alpha ct} = e^{-i\alpha c_r t} e^{+\alpha c_i t}, \tag{140}$$

  with temporal instability if $c_i > 0$ and temporal stability if $c_i < 0$. (Note the different convention from previous sections, which had $\exp(st)$ without an i in the exponent, giving instability if $s_r > 0$.)

- Spatial instability – Here we assume that (ω, β) are real, but allow the streamwise wavenumber to be complex:

$$e^{i\alpha x} = e^{i\alpha_\mathrm{r} x} e^{-\alpha_\mathrm{i} x}. \tag{141}$$

  This gives spatial instability if $\alpha_\mathrm{i} < 0$ and spatial stability if $\alpha_\mathrm{i} > 0$.

In what follows, we will consider only the temporal stability problem.

4.1.4 Linearise the equations

Considering temporal the stability problem with $\omega = \alpha c$ and (α, β) real, we now substitute the perturbed form (138, 139) into the governing equations (136, 137), and expand in powers of δ. At $O(\delta)$ we then get the linearised equations

Continuity

$$i(\alpha\tilde{u} + \beta\tilde{w}) + \tilde{v}' = 0, \tag{142}$$

Momentum balance

$$i\alpha(u_\mathrm{B} - c)\tilde{u} + \tilde{v}u_\mathrm{B}'(y) = -i\alpha\tilde{p} + \frac{1}{Re}(D^2 - k^2)\tilde{u}, \tag{143}$$

$$i\alpha(u_\mathrm{B} - c)\tilde{v} = -\tilde{p}'(y) + \frac{1}{Re}(D^2 - k^2)\tilde{v}, \tag{144}$$

$$i\alpha(u_\mathrm{B} - c)\tilde{w} = -i\beta\tilde{p} + \frac{1}{Re}(D^2 - k^2)\tilde{w}, \tag{145}$$

$$\text{where} \quad D \equiv \frac{d}{dy} \quad \text{and} \quad k^2 = \alpha^2 + \beta^2. \tag{146}$$

To simplify the problem, our strategy will be first to eliminate $\tilde{u}, \tilde{w}, \tilde{p}$ to leave a single equation in $\tilde{v}$. This can be used finally to determine the linear stability (or instability) of the base flow.

Taking $i\alpha(143) + i\beta(145)$ we get

$$i\alpha(u_{\mathrm{B}} - c)[i\alpha\tilde{u} + i\beta\tilde{w}] + i\alpha\tilde{v}u_{\mathrm{B}}' = k^2\tilde{p} + \frac{1}{Re}(D^2 - k^2)[i\alpha\tilde{u} + i\beta\tilde{w}]. \tag{147}$$

Using the continuity equation (142), we can replace $[i\alpha\tilde{u} + i\beta\tilde{w}]$ with $-D\tilde{v}$ to get:

$$-i\alpha(u_{\mathrm{B}} - c)D\tilde{v} + i\alpha\tilde{v}u_{\mathrm{B}}' = k^2\tilde{p} - \frac{1}{Re}(D^2 - k^2)D\tilde{v}. \tag{148}$$

Thus we have eliminated $\tilde{u}, \tilde{w}$ to leave (144, 148) in $\tilde{v}, \tilde{p}$. Finally we eliminate $\tilde{v}$ as follows. Operating across (148) with D we get

$$-i\alpha(u_{\mathrm{B}} - c)D^2\tilde{v} + i\alpha\tilde{v}u_{\mathrm{B}}'' = k^2 D\tilde{p} - \frac{1}{Re}(D^2 - k^2)D^2\tilde{v}. \tag{149}$$

from which we have cancelled equal and opposite terms $\pm i\alpha u_{\mathrm{B}}' D\tilde{v}$. (Do this as an exercise.) From (144) we have

$$D\tilde{p} = -i\alpha(u_{\mathrm{B}} - c)\tilde{v} + \frac{1}{Re}(D^2 - k^2)\tilde{v} \tag{150}$$

which can be substituted into (149) to give

$$-i\alpha(u_{\mathrm{B}} - c)D^2\tilde{v} + i\alpha\tilde{v}u_{\mathrm{B}}'' = -i\alpha k^2(u_{\mathrm{B}} - c)\tilde{v} + \frac{k^2}{Re}(D^2 - k^2)\tilde{v} - \frac{1}{Re}(D^2 - k^2)D^2\tilde{v}. \tag{151}$$

This tidies up to

$$\frac{1}{Re}(D^2 - k^2)(D^2 - k^2)\tilde{v} - i\alpha\left[(u_{\mathrm{B}} - c)D^2\tilde{v} + k^2(-1)(u_{\mathrm{B}} - c)\tilde{v} - u_{\mathrm{B}}''\tilde{v}\right] = 0, \tag{152}$$

and finally

$$(D^2 - k^2)^2\tilde{v} - i\alpha Re\left[(u_{\mathrm{B}} - c)(D^2 - k^2)\tilde{v} - u_{\mathrm{B}}''\tilde{v}\right] = 0. \tag{153}$$

This is the **Orr-Somerfeld equation**, governing the linear stability/instability of planar shear flow with respect to 3D perturbations, for viscous fluids.

4.1.5 Boundary conditions revisited

In Sec. 4.1.1 above we specified the boundary conditions at the walls $y = y_1, y_2$ to be no permeation and no slip. We now translate these into conditions for the function $\tilde{v}$ in the Orr-Somerfeld equation (153).

For the perturbation $(\tilde{u}, \tilde{v}, \tilde{w})$ we have

$$\tilde{u} = \tilde{w} = 0 \quad \text{on} \quad y = y_1, y_2 \quad \text{(no slip)}. \tag{154}$$

Via the continuity equation (136), this is seen to be equivalent to

$$D\tilde{v} = 0 \quad \text{on} \quad y = y_1, y_2 \quad \text{(no slip)}. \tag{155}$$

We also have

$$\tilde{v} = 0 \quad \text{on} \quad y = y_1, y_2 \quad \text{(no permeation)}. \tag{156}$$

Together (155, 156) give the 4 boundary conditions on $\hat{v}$ needed to fully specify the solution to the 4th order Orr-Somerfeld equation (153).

4.2 Squire's theorem

As with all linear stability problems, the Orr-Somerfeld equation (OSE) with boundary conditions (155, 156) has the form of an eigenvalue problem

$$c = c(\alpha, k, Re; u_{\mathrm{B}}(y)). \tag{157}$$

For any given α, k, Re this can be used to determine whether the base flow $u_{\mathrm{B}}(y)$ is linearly stable, $c_i < 0$, or linearly unstable, $c_i > 0$. (Recall that we are considering modes such that the perturbation develops as $\exp(-i\alpha ct) = \exp(\alpha c_{\mathrm{i}} t)\exp(-i\alpha c_{\mathrm{r}} t)$.)

Often, however, we are only interested in the instability that appears first as the control parameter Re is increased. In this case, Squire's theorem tells us that we need only consider 2D disturbances:

Squire's theorem — If a growing 3D disturbance can be found at a given Reynolds number, then a growing 2D disturbance exists at a lower Reynolds number.

This can be seen as follows. Consider a base state $u_{\mathrm{B}}(y)$. Imagine a growing 3D disturbance to this base state at Reynolds number $Re_{3\mathrm{D}}$, with wavenumbers $\alpha_{3\mathrm{D}}$, $\beta_{3\mathrm{D}}$, and $k_{3\mathrm{D}}^2 = \alpha_{3\mathrm{D}}^2 + \beta_{3\mathrm{D}}^2$. This corresponds to a solution $c, \tilde{v}$ with $c_i > 0$ of the 3D OSE

$$\frac{1}{i\alpha_{3\mathrm{D}} Re_{3\mathrm{D}}}(D^2 - k_{3\mathrm{D}}^2)^2\,\tilde{v} = \left[(u_{\mathrm{B}} - c)(D^2 - k_{3\mathrm{D}}^2) - D^2 u_{\mathrm{B}}\right]\tilde{v}. \tag{158}$$

Now consider a 2D disturbance at a Reynolds number $Re_{2\mathrm{D}}$. This has $\beta = 0$, $k_{2\mathrm{D}} = \alpha_{2\mathrm{D}}$ and must satisfy the 2D OSE

$$\frac{1}{i\alpha_{2\mathrm{D}} Re_{2\mathrm{D}}}(D^2 - \alpha_{2\mathrm{D}}^2)^2\,\tilde{v} = \left[(u_{\mathrm{B}} - c)(D^2 - \alpha_{2\mathrm{D}}^2) - D^2 u_{\mathrm{B}}\right]\tilde{v}. \tag{159}$$

For values $\alpha_{2\mathrm{D}} = k_{3\mathrm{D}}$ and $Re_{2\mathrm{D}} = \alpha_{3\mathrm{D}} Re_{3\mathrm{D}}/\alpha_{2\mathrm{D}} = \alpha_{3\mathrm{D}} Re_{3\mathrm{D}}/k_{3\mathrm{D}}$, this 2D OSE has the form

$$\frac{1}{i\alpha_{3\mathrm{D}} Re_{3\mathrm{D}}}(D^2 - k_{3\mathrm{D}}^2)^2\,\tilde{v} = \left[(u_{\mathrm{B}} - c)(D^2 - k_{3\mathrm{D}}^2) - D^2 u_{\mathrm{B}}\right]\tilde{v}, \tag{160}$$

which is exactly the same as (158). It must therefore have the same growing solution $c, \tilde{v}$ with $c_i > 0$.

Therefore, corresponding to the growing 3D disturbance at $Re_{3\mathrm{D}}$ with $\alpha_{3\mathrm{D}}, \beta_{3\mathrm{D}}, k_{3\mathrm{D}}^2 = \alpha_{3\mathrm{D}}^2 + \beta_{3\mathrm{D}}^2$, there exists a growing 2D disturbance at $Re_{2\mathrm{D}}$ with $\alpha_{2\mathrm{D}} = k_{3\mathrm{D}}$.

It remains finally to show that $Re_{2\mathrm{D}} \leq Re_{3\mathrm{D}}$. This is done as follows. Since $k_{3\mathrm{D}}^2 = \alpha_{3\mathrm{D}}^2 + \beta_{3\mathrm{D}}^2$, we have $k_{3\mathrm{D}} \geq \alpha_{3\mathrm{D}}$ and so $Re_{2\mathrm{D}} \leq Re_{3\mathrm{D}}$. Hence, the 2D disturbance grows at the lower Reynolds number $Re_{2\mathrm{D}} \leq Re_{3\mathrm{D}}$, as originally claimed.

4.3 The inviscid theory

We now consider the limit of inviscid flows, $Re \to \infty$. We start in Sec. 4.3.1 by introducing Rayleigh's equation, which is the counterpart of the Orr-Somerfeld equation in this limit. We then use Rayleigh's equation to derive Rayleigh's inflexion point theorem, Sec. 4.3.2, which tells us that a necessary condition for (inviscid) instability is the existence of an inflexion point $D^2 u_{\mathrm{B}}(y) = 0$ somewhere in the flow domain $y_1 < y < y_2$.

4.3.1 Rayleigh's equation

Fixing y and α and letting $Re \to \infty$, the Orr-Somerfeld equation (153) becomes

$$(u_{\mathrm{B}}(y) - c)(D^2 - \alpha^2)\tilde{v} - (D^2 u_{\mathrm{B}})\tilde{v} = 0. \tag{161}$$

This is **Rayleigh's equation**. In contrast to the original Orr-Somerfeld equation, which contained derivatives $D^4\tilde{v}$ of fourth order, Rayleigh's equation contains only derivatives $D^2\tilde{v}$ that are at most of second order. It can therefore in general only be solved subject to two boundary conditions:

$$\tilde{v} = 0 \quad \text{on} \quad y = y_1, y_2 \quad \text{(no permeation at each wall).} \tag{162}$$

In this way, the condition of no slip is (apparently) discarded. Refer to any textbook on boundary layers if you would like to know more about this. The limit $Re \to \infty$ is said to be *singular*, since $1/Re$ multiplies the highest order derivative in (153).

A property of Rayleigh's equation

• If c is an eigenvalue of Rayleigh's equation, then so is its complex conjugate $\bar{c}$.

This can be seen as follows. Suppose that $\tilde{v}(y; c)$ is a non-trivial solution of Rayleigh's equation:

$$(u_{\mathrm{B}} - c)(D^2 - \alpha^2)\tilde{v} - (D^2 u_{\mathrm{B}})\tilde{v} = 0 \quad \text{with} \quad \tilde{v}(y_1; c) = \tilde{v}(y_2; c) = 0. \tag{163}$$

Taking the complex conjugate of this, we get

$$(u_{\mathrm{B}} - \bar{c})(D^2 - \alpha^2)\bar{v} - (D^2 u_{\mathrm{B}})\bar{v} = 0 \quad \text{with} \quad \bar{v}(y_1; \bar{c}) = \bar{v}(y_2; \bar{c}) = 0, \tag{164}$$

where $\bar{v}$ is the complex conjugate of $\tilde{v}$. Thus, if $\tilde{v}(y; c)$ is a solution of Rayleigh's equation, then so is $\bar{v}(y; \bar{c})$ as stated above.

This property is important because it tells us that if *any* complex eigenvalue can be found then the base flow must be unstable, because one of c and $\bar{c}$ will have an imaginary part greater than zero, $c_i > 0$. (Recall that we are considering modes such that the perturbation develops as $\exp(-i\alpha c t) = \exp(\alpha c_{\mathrm{i}} t)\exp(-i\alpha c_{\mathrm{r}} t)$.) We will now make use of this in our derivation of Rayleigh's inflexion point theorem.

4.3.2 Rayleigh's inflexion point theorem

Suppose that u_B and Du_B are continuous in $y_1 < y < y_2$. Rayleigh's inflexion point theorem then states that a necessary (though not sufficient) condition for inviscid instability is that the base state possesses an inflexion point $D^2 u_\mathrm{B} = 0$ somewhere in the domain $y_1 < y < y_2$. If a base state lacks an inflexion point, therefore, we can conclude it to be stable, for inviscid fluids.

This can be seen as follows. Consider Rayleigh's equation in the following form:

$$D^2\tilde{v} - \left(\alpha^2 + \frac{D^2u_\mathrm{B}}{u_\mathrm{B} - c}\right)\tilde{v} = 0. \tag{165}$$

As usual $c = c_\mathrm{r} + ic_\mathrm{i}$. Our strategy will be initially to *suppose* that the flow is unstable, $c_\mathrm{i} > 0$, and then to prove that an inflexion point $D^2u_\mathrm{B} = 0$ must exist for this to be so.

Pre-multiplying (165) across by $\bar{v}$, the complex conjugate of $\tilde{v}$, and integrating from y_1 to y_2, we get

$$\int_{y_1}^{y_2} \bar{v} D^2 \tilde{v} dy - \int_{y_1}^{y_2} \left(\alpha^2 + \frac{D^2u_\mathrm{B}}{u_\mathrm{B} - c}\right) |\tilde{v}|^2 \, dy = 0, \tag{166}$$

with $|\tilde{v}|^2 = \tilde{v}\bar{v}$. Integrating the first term by parts we get

$$[D\tilde{v}.\bar{v}]_{y_1}^{y_2} - \int_{y_1}^{y_2} D\tilde{v}.D\bar{v} \, dy - \int_{y_1}^{y_2} \left(\alpha^2 + \frac{D^2u_\mathrm{B}}{u_\mathrm{B} - c}\right) |\tilde{v}|^2 \, dy = 0. \tag{167}$$

Using the boundary condition $\bar{v}(y_1) = \bar{v}(y_2) = 0$, we see that the first term equals zero, leaving

$$-\int_{y_1}^{y_2} |D\tilde{v}|^2 dy - \int_{y_1}^{y_2} \left(\alpha^2 + \frac{D^2u_\mathrm{B}}{u_\mathrm{B} - c}\right) |\tilde{v}|^2 \, dy = 0. \tag{168}$$

Multiplying both the numerator and denominator of the second term in the brackets by $u_\mathrm{B} - \bar{c}$, we get

$$-\int_{y_1}^{y_2} |D\tilde{v}|^2 dy - \int_{y_1}^{y_2} \left(\alpha^2 + \frac{D^2u_\mathrm{B}.(u_\mathrm{B} - \bar{c})}{|u_\mathrm{B} - c|^2}\right) |\tilde{v}|^2 \, dy = 0. \tag{169}$$

The imaginary part of this equation is

$$-c_\mathrm{i} \int_{y_1}^{y_2} \frac{D^2u_\mathrm{B}|\tilde{v}|^2}{|u_\mathrm{B} - c|^2} dy = 0. \tag{170}$$

Let us denote the integral in this expression by I. For (170) to be satisfied, we must clearly have either $c_\mathrm{i} = 0$ or $I = 0$. We have already assumed that $c_\mathrm{i} > 0$, corresponding to unstable flow. Therefore, we must have $I = 0$. Examining the various components of I's integrand, we see that $|\tilde{v}|^2 > 0$ and $|u_\mathrm{B} - c|^2 > 0$. For $I = 0$, therefore, D^2u_B must change sign somewhere in the domain (y_1, y_2).

Finally, then, we can conclude that

- a necessary condition for inviscid instability is the presence of an inflexion point;
- the absence of an inflexion point necessarily confers (inviscid) stability.

4.3.3 Other stability results

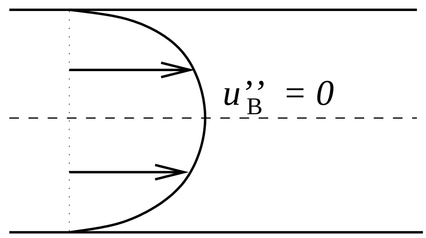
Here we state without proof some other results for the stability of inviscid flows.

• **Fjortoft's theorem** A necessary condition for instability is that

$$u_B''(u_B - u_c) < 0 \quad \text{somewhere in the fluid,} \tag{171}$$

where u_c is the flow speed at the inflexion point (*i.e.* $u_c = u_B(y_c)$ with $u_B''(y_c) = 0$).

• **Tollmien's result** For a *symmetrical* profile in a channel, or for a *boundary layer*, the existence of an inflexion point $u_B''(y_c) = 0$ is not only necessary but also a sufficient condition for instability. The inviscid flow sketched as follows is thus linearly unstable.



• **Howard's semi-circle theorem** All unstable waves have $c = c_r + ic_i$ satisfying

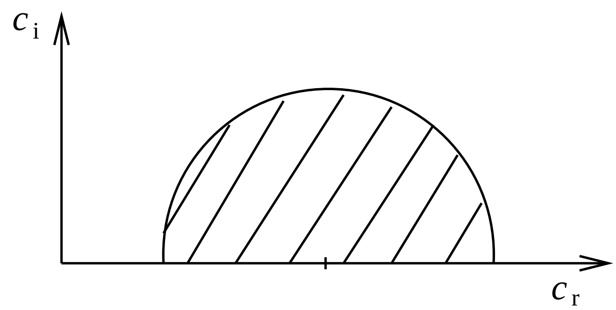
$$\left[c_r - \frac{1}{2}(u_{max} - u_{min})\right]^2 + c_i^2 \leq \frac{1}{2}(u_{max} - u_{min})^2. \tag{172}$$

where

$$u_{max} = \max u_B(y) \tag{173}$$

$$u_{min} = \min u_B(y) \tag{174}$$

Thus, all unstable modes lie in the shaded semi-circle sketched below, centred on $c_r = \frac{1}{2}(u_{max} + u_{min})$, $c_i = 0$ and of radius $\frac{1}{4}(u_{max} - u_{min})$.



4.4 Plane Poiseuille flow

In Secs. 4.1 to 4.3 above we considered a general base state $u_{\mathrm{B}}(y)$. We now turn to the concrete example of plane Poiseuille flow:

$$u_{\mathrm{B}} = (1 - y^2) \quad \text{for} \quad -1 \leq y \leq 1, \tag{175}$$

which corresponds to flow between two infinite parallel planes at $y_1 = -1, y_2 = +1$, driven by a pressure gradient $\partial_x p_{\mathrm{B}} = -2/Re$. For this we have

$$D^2 u_{\mathrm{B}} = -2 \tag{176}$$

everywhere in the flow domain, with no inflexion point. By Rayleigh's inflexion point theorem, therefore, we can conclude the flow to be linearly stable in the inviscid limit $Re \to \infty$.

For finite Re, the Orr-Somerfeld equation must be solved numerically. The results are sketched in Fig. 4.4. Several features are to be noted, as follows. For any $Re > R_c \approx 5772$, no matter how large, we have linear instability for a band of wavenumbers $\alpha_{\mathrm{L}}(Re) < \alpha < \alpha_{\mathrm{U}}(Re)$. However both α_{L} and $\alpha_{\mathrm{U}} \to 0$ as $Re \to \infty$, with $\alpha_{\mathrm{L}} \sim Re^{-1/7}$ and $\alpha_{\mathrm{U}} \sim Re^{-1/11}$. So for any fixed α^* the flow is stable in the inviscid limit $Re \to \infty$, consistent with Rayleigh's inflexion point theorem.