4 Exact laminar boundary layer solutions

4.1 Boundary layer on a flat plate (Blasius 1908)

In Sec. 3, we derived the boundary layer equations for 2D incompressible flow of constant viscosity past a weakly curved or flat surface. We now solve them for the case of a flat plate of length L, set at zero angle of incidence to a uniform stream of velocity U. (Recall Fig. 5.) Looking back at the non-dimensional form of the BL equations 50, 51 and 56, we see that the general form of the BL solution is

$$u', \;\; v', \;\; \Psi' = \text{functions}(x', y'; u'_e(x')). \tag{79}$$

We assume that the plate is thin enough not disturb the uniform exterior flow, so that

$$u_e = U = \text{constant}, \;\; i.e., \;\; u'_e = 1. \tag{80}$$

We then have

$$u', \;\; v', \;\; \Psi' = \text{functions}(x', y'). \tag{81}$$

In dimensional form

$$\frac{u}{U}, \;\; \frac{v}{U}Re^{1/2}, \;\; \frac{\Psi}{\sqrt{\nu UL}} = \text{functions}(\frac{x}{L}, \frac{y}{L}Re^{1/2}) \;\; \text{in which} \;\; Re = \frac{UL}{\nu}. \tag{82}$$

This can be simplified even further using the concept of **self-similarity**, which we motivate as follows. Looking again at the BL equations, we see that the flow profile at any $x = x_1$ is only affected by advection from upstream $x < x_1$, and by viscous diffusion in the y direction. The flow at x_1 therefore only knows about the previous history, $x < x_1$, and not about the future, $x > x_1$. Mathematically, this is due to the boundary layer equations being **parabolic**. In direct consequence, at any x the profile cannot know how far the plate extends into the future, L. We must therefore replace L by another length scale. The only candidate for this is x itself, so we now have

$$\frac{u}{U}, \;\; \frac{v}{U}Re_x^{1/2}, \frac{\Psi}{\sqrt{\nu Ux}} = \text{functions}(\frac{y}{x}Re_x^{1/2}) = \text{functions}(\frac{y}{\delta(x)}), \tag{83}$$

in which

$$Re_x = \frac{Ux}{\nu} \;\; \text{and} \;\; \delta(x) = \frac{x}{Re_x^{1/2}} = \sqrt{\frac{x\nu}{U}}. \tag{84}$$

The flow profile thus varies with the horizontal position x only via a rescaling of y by the length $\delta(x) \propto x^{1/2}$: it is self-similar, Fig. 8. We identify $\delta(x)$ as the length scale that sets the local boundary layer thickness. You might recognise this as a diffusive process: information diffuses a distance $\delta \propto t^{1/2}$ away from the plate due to viscous effects in time t, during which it has been advected along the plate a distance $x \propto t$.

Given that u/U, *etc.*, can depend only on the single scaling variable $yRe_x^{1/2}/x = y(U/\nu x)^{1/2}$, it must be possible to recast the momentum equation as an ordinary differential equation with $y(U/\nu x)^{1/2}$ as the only independent variable, rather than a partial differential equation in x and y. (By using a stream function, we will not need to consider the continuity equation explicitly.) We do this now by changing variables to

$$\eta = \left(\frac{U}{2\nu x}\right)^{1/2} y, \quad \text{and} \quad \xi = x, \tag{85}$$

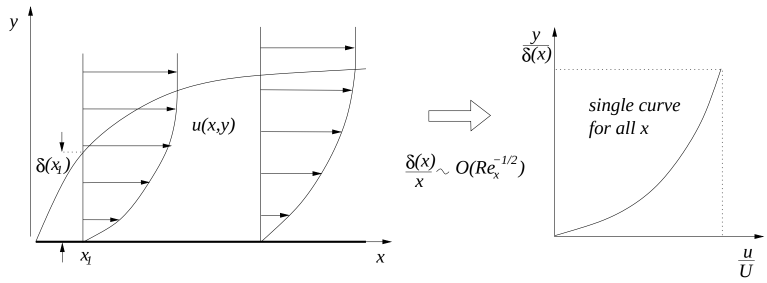

Figure 8: Self similarity of the boundary layer solution.

and expressing the scaled stream function as a function f of η

$$\frac{\Psi}{\sqrt{2\nu U x}} = f(\eta). \tag{86}$$

The factors of 2 are purely for convenience. Writing $\xi = x$ is also a convenience to avoid confusion between old (x, y) and new (ξ, η) variables. In any case, the final form of the momentum equation will depend only on η and not on ξ, for the reasons just given.

In terms of the new variables (ξ, η), the partial derivatives

$$\left(\frac{\partial}{\partial x}\right)_y = \left(\frac{\partial}{\partial \xi}\right)_\eta \left(\frac{\partial \xi}{\partial x}\right)_y + \left(\frac{\partial}{\partial \eta}\right)_\xi \left(\frac{\partial \eta}{\partial x}\right)_y = \frac{\partial}{\partial \xi} - \frac{\eta}{2\xi}\frac{\partial}{\partial \eta} \tag{87}$$

and

$$\left(\frac{\partial}{\partial y}\right)_x = \left(\frac{\partial}{\partial \xi}\right)_\eta \left(\frac{\partial \xi}{\partial y}\right)_x + \left(\frac{\partial}{\partial \eta}\right)_\xi \left(\frac{\partial \eta}{\partial y}\right)_x = \left(\frac{U}{2\nu\xi}\right)^{1/2} \frac{\partial}{\partial \eta}. \tag{88}$$

(Check these as an exercise.) The velocity components, given in terms of the stream function by Eqns. 53, then become:

$$u = \frac{\partial \Psi}{\partial y} = \left(\frac{U}{2\nu\xi}\right)^{1/2} \frac{\partial}{\partial \eta}\{(2\nu U\xi)^{1/2} f(\eta)\} = U\frac{df}{d\eta}, \tag{89}$$

and

$$-v = \frac{\partial \Psi}{\partial x} = \left(\frac{\partial}{\partial \xi} - \frac{\eta}{2\xi}\frac{\partial}{\partial \eta}\right)\{(2\nu U\xi)^{1/2} f(\eta)\} = (2\nu U)^{1/2}\left\{\frac{f}{2\xi^{1/2}} - \frac{\eta}{2\xi}\xi^{1/2}\frac{df}{d\eta}\right\}. \tag{90}$$

The convective operator is then

$$\begin{aligned} u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} &= Uf'\left(\frac{\partial}{\partial \xi} - \frac{\eta}{2\xi}\frac{\partial}{\partial \eta}\right) - \left(\frac{\nu U}{2\xi}\right)^{1/2}(f - \eta f')\left(\frac{U}{2\nu\xi}\right)^{1/2}\frac{\partial}{\partial \eta} \\ &= U\left(f'\frac{\partial}{\partial \xi} - \frac{f}{2\xi}\frac{\partial}{\partial \eta}\right). \end{aligned} \tag{91}$$

Inserting these into the momentum equation 70 with $u_e = U =$ constant, we get

$$U\left(f'\frac{\partial}{\partial \xi} - \frac{f}{2\xi}\frac{\partial}{\partial \eta}\right)\{Uf'\} = \nu\left(\frac{U}{2\nu\xi}\right)^{1/2}\frac{\partial}{\partial \eta}\left\{\left(\frac{U}{2\nu\xi}\right)^{1/2}\frac{\partial}{\partial \eta}\{Uf'\}\right\}. \tag{92}$$

Because f' is a function of η only, we can ignore the term in $\partial/\partial\xi = 0$ on the LHS. Dividing the remaining terms across by the constant $U^2/2\xi$, and tidying up, we obtain

$$ff'' + f''' = 0. \tag{93}$$

(Check this as an exercise.) This is the final simplified form of the momentum equation, Eqn. 73, expressed in terms of the scaled stream function f. The term in f''' comes from the viscous terms on the RHS of the original equation; ff'' from advection.
We recall that the BCs for the boundary layer equations are

- $u = v = 0$ at the solid surface $y = 0$.
- $u \to u_e(x)$ at the exterior edge $y \to \infty$. For the flat plate $u_e = U$.

We transcribe these into BCs for Eqn. 93 by recalling that $\eta \propto y/\sqrt{x}$ and using Eqn. 89 ($u = Uf'$) and Eqn. 90 ($v \propto f - \eta f'$), to get:

- $f' = 0$, $f = 0$ at the solid surface $\eta = 0$.
- $f' \to 1$ at the exterior edge $\eta \to \infty$.

There are thus three BCs, corresponding to the order of the differential equation.

Equations such as 93 generally can't be solved analytically. They must be solved numerically. The usual method is to start at $\eta = 0$ and march along the η axis, numerically generating successive values of f. If all three BCs were specified at $\eta = 0$ (say if we knew $f(0)$, $f'(0)$ and $f''(0)$) this would be simple: We would only need to march once, because the entire solution $f(\eta)$ would be specified by these three conditions. However we only know two BCs at $\eta = 0$: $f(0) = 0$ and $f'(0) = 0$. The other is at $\eta \to \infty$. To resolve this apparent impasse, we *guess* the third boundary value $f''(0) = g$ (for guess), and generate (by marching) the correspondingly specified $f_g(\eta)$. Of course this will not, in general, satisfy the third (actual) BC, $f'(\infty) = 0$. We must therefore systematically iterate g until the $f_g(\eta)$ it generates does obey $f'(\infty) = 0$. This is known as the "shooting" method for solving differential equations, Fig. 9.

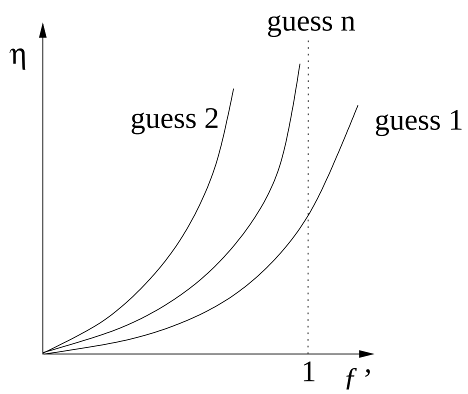

Figure 9: The shooting method for solving differential equations.

In the case of the flat plate (but not for any other BL that we shall encounter), this process can actually be simplified considerably by a special property of the differential equation 93 called the affine transformation. We put

$$a\eta = X, \quad f(\eta) = aF(X), \tag{94}$$

where a is an arbitrary constant, so that

$$\frac{df}{d\eta} = a\frac{dF}{dX}\frac{dX}{d\eta} = a^2F', \quad \frac{d^2f}{d\eta^2} = \frac{d}{dX}(a^2F')\frac{dX}{d\eta} = a^3F'', \quad \frac{d^3f}{d\eta^3} = a^4F'''. \tag{95}$$

(Dashes now denote differentiation with respect to X.) Eqn. 93 then becomes

$$F''' + FF'' = 0, \tag{96}$$

which is *the same* as the original Eqn. 93, but now with BCs

$$F(0) = F'(0) = 0, \quad F'(\infty) = 1/a^2. \tag{97}$$

The point is that we can solve this for $F(X)$ using *any* value of a that we choose, then back-out the solution $f(x)$ by reversing the transformation 94. Put another way, we can choose any $F''(0)$ (let's set $F''(0) = 1$ to be definite), find the corresponding $F_{g=1}(X)$ by the usual marching method, see what $F'_{g=1}(\infty)$ it has, calculate the corresponding $a = 1/\sqrt{F'_{g=1}(\infty)}$, then reverse the transformation to find $f(\eta)$. Using this trick, we only need to do the numerical marching process once: we have removed the need to iterate!

Having obtained the numerical solution, what do we want to extract from it? The most important thing is usually a value for the surface shear stress, *i.e.*

$$(\Pi_{xy})_{y=0} = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu U\left(\frac{U}{2\nu x}\right)^{1/2}\left(\frac{\partial f'}{\partial \eta}\right)_{\eta=0} = \mu U\left(\frac{U}{2\nu x}\right)^{1/2} f''(0) \tag{98}$$

or

$$c_f Re_x^{1/2} = \frac{2(\Pi_{xy})_{y=0}}{\rho U^2}\left(\frac{Ux}{\nu}\right)^{1/2} = \frac{2\nu}{U}\left(\frac{Ux}{\nu}\frac{U}{2\nu x}\right)^{1/2} f''(0) = \sqrt{2}f''(0), \tag{99}$$

Notice that the x dependence has been absorbed into the definition of Re. From the affine transformation we know that

$$f''(0) = a^3F''(0) = a^3 = (F'_{g=1}(\infty))^{-3/2}. \tag{100}$$

Using the numerically calculated value for $F'_{g=1}(\infty)$, we get

$$f''(0) = 0.46960, \quad c_f Re_x^{1/2} = 0.6641. \tag{101}$$

Another important quantity is the thickness of the boundary layer. How do we define this? Looking back at the definition of the scaling variable η in Eqn. 85, we see that the characteristic length scale on which the profile attains the free stream is $\sqrt{2\nu x/U}$. (This equals the $\delta(x)$ of Eqn. 84 to within an unimportant prefactor.) It therefore sets the typical scale of the BL's thickness. Strictly, however, the profile only truly attains the free stream in the limit $\eta \to \infty$ (where $f' \to 1$.) This gives $y \to \infty$ too, which is clearly not a practical measure. One way to resolve this is to argue that *for all practical purposes* the free stream is reached once $f' = 1.000$ to 4 figure accuracy. This turns out to be when $\eta = \eta_\delta \approx 5$, giving a length

$$\delta = \eta_\delta\left(\frac{2\nu x}{U}\right)^{1/2} \quad \text{or} \quad \frac{\delta}{x}Re_x^{1/2} = \sqrt{2}\eta_\delta. \tag{102}$$

Strictly, though, for f' to truly attain 1, η_δ is infinite.

The quantity just defined is obviously arbitrary to the extent that η_δ depends on the value of smallness imposed on $f' - 1$. Two measures that avoid this difficulty are the displacement and momentum thicknesses defined by Eqns. 77 and 78. In the case of the flat plate, $u_e = U$, so

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy, \quad \theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy. \tag{103}$$

As an exercise, show using the usual relations $\eta = y\sqrt{U/2\nu x}$ and $u = f'U$ that

$$\frac{\delta^*}{x} Re_x^{1/2} = \sqrt{2} \lim_{\eta\to\infty} (\eta - f) = 1.7208, \tag{104}$$

$$\frac{\theta}{x} Re_x^{1/2} = \sqrt{2} f''(0) = 0.6641. \tag{105}$$

(The final numbers have to be extracted from the numerical solution and are given for interest only.) Thus, both δ^* and θ show the same behaviour as δ with respect to x, ν and U, but have finite values. The main point to note is the "diffusive" scaling of thickness $\propto \sqrt{x}$, as discussed above.