4.4 The two-dimensional jet at high Reynolds number

Boundary layer theory is not restricted to flow near a solid surface. It applies to any thin layer in which viscous forces dominate. In this section, we use it to study a steady jet emerging from a slit into a surrounding fluid. In practice, jets become turbulent at high Re, but we study the laminar case because the turbulent analysis builds on this.

We assume planar flow, corresponding to a long slit perpendicular to the paper in Fig. 15. For simplicity, we assume the slit to be infinitely thin (which actually requires an infinitely large velocity at the slit to retain a finite flow volume). The emerging jet carries with it some of the surrounding fluid. This was originally at rest, but is set in motion by the jet due to viscous friction. Viscous effects therefore are important in the jet: this is how it spreads out with downstream distance from the slit. For moderate x values the jet remains slender, there are steep gradients in the y direction, and the usual boundary layer equations apply.

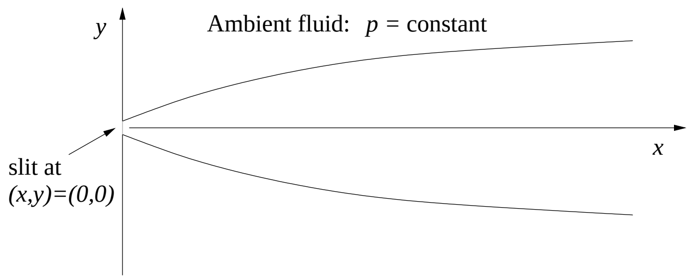

Figure 15: The 2D jet.

4.4.1 Governing equations

As just remarked, the jet obeys the usual boundary layer equations:
Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{133}$$

Momentum

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{134}$$

with $u_e(x)$ the velocity on the exterior edge given by inviscid theory. In this case the (far) surrounding fluid is at rest, so $u_e = 0$ and the momentum equation becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{135}$$

4.4.2 Boundary conditions

- As $y \to \infty$ we have the exterior velocity $u \to u_e(x)$ with $u_e = 0$, as just noted.
- We assume that the jet is symmetric about $y = 0$, Fig. 16, so that
  - The $x-$velocity is an even function, $u(y) = u(-y)$, giving $(\partial u/\partial y)_{y=0} = 0$
  - The $y-$velocity is an odd function, $v(y) = -v(-y)$, giving $v(0) = 0$.

4.4.3 Constant momentum flux

A useful property of the solution, to be utilised below, is that the total flux of $x-$momentum, integrated on y at any station x, is a constant, independent of x:

$$M = \rho \int_{-\infty}^{+\infty} u^2 dy = \text{constant}. \tag{136}$$

This is constructed by realising that the mass per unit time crossing a unit surface area with normal in the x direction is ρu. (Recall Sec. 1.2.1.) This carries with it an $x-$momentum ρu^2, Sec. 1.3.

Physically, the fact that M is constant follows because there is no pressure gradient, $dp/dx = 0$. Mathematically, we prove it as follows. The momentum equation 135 can be written as

$$u\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}(uv) - u\frac{\partial v}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}. \tag{137}$$

Using the continuity equation

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}, \tag{138}$$

this can be rewritten as

$$u\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}(uv) + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{139}$$

i.e.

$$\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) = \nu\frac{\partial^2 u}{\partial y^2}. \tag{140}$$

Integrating from $y = -\infty$ to $y = +\infty$ at a given horizontal position x, and taking into account the symmetry[5] of the jet, we get

$$\frac{d}{dx}\int_{-\infty}^{+\infty} u^2 dy + 2[vu]_0^{+\infty} = 2\nu\left[\frac{\partial u}{\partial y}\right]_0^{+\infty}. \tag{141}$$

Using the BCs $v(0) = 0$, $\frac{\partial u}{\partial y}(y = 0) = 0$, $u(\infty) = 0$, from which $\frac{\partial u}{\partial y}(y = \infty) = 0$, we we get

$$\frac{d}{dx}\int_{-\infty}^{+\infty} u^2 dy = 0, \tag{142}$$

which proves that the total $x-$momentum flux M is constant, Eqn. 136.

We assume that M is a known parameter, controlled from the outside. Its dimensions are $[M] = \rho U^2 L$, where U and L are characteristic velocity and length scales. We define a Reynolds number based on it:

$$Re = \frac{UL}{\nu} = \left(\frac{M}{\rho L}\right)^{1/2}\frac{L}{\nu} = \left(\frac{ML}{\rho \nu^2}\right)^{1/2}. \tag{143}$$

As the jet spreads outwards with increasing x, the velocity at its centre decreases to maintain constant M, Fig. 16.

[5] u is an even function of y; v is odd; uv is odd; $\partial\,[uv]\,/\partial y$ is even; $\partial^2 u/\partial y^2$ is even.

4.4.4 Self-similar solutions

In this section, we solve the governing equations 133 and 135 subject to the BCs specified in Sec. 4.4.2, for a given imposed momentum flux M. As usual, we utilise the stream function defined by Eqn. 72 so that continuity, Eqn. 133 is automatically satisfied.

Similarity Variables

We seek a self-similar solution in the form

$$\Psi \sim x^p f(\eta) \;\text{ with }\; \eta \sim \frac{y}{x^q} \tag{144}$$

where x^q sets the scale of the jet thickness at a given x. Constant prefactors in 144 will be specified later on. This corresponds to a horizontal velocity component

$$u = \frac{\partial \Psi}{\partial y} \sim x^{p-q} f'(\eta), \tag{145}$$

from which we see that x^{p-q} gives the local velocity scale[6].

To determine the two unknown constants p and q, we need two conditions:

1. Constant $x-$momentum flux

$$\int u^2 dy = \text{constant}. \tag{146}$$

Using $u^2 \sim x^{2(p-q)} F(\eta)$ and $dy \sim x^q d\eta$ we get

$$\begin{aligned} x^{2(p-q)+q} &\sim x^0, \\ 2p - q &= 0. \end{aligned} \tag{147}$$

2. Inertial and viscous forces must balance in the momentum equation

$$u\frac{\partial u}{\partial x} \sim \nu \frac{\partial^2 u}{\partial y^2} \tag{148}$$

i.e.,

$$\frac{u^2}{x} \sim \frac{u}{y^2}. \tag{149}$$

Using $u \sim x^{p-q} f'(\eta)$ and $y \sim \eta x^q$ we get

$$\begin{aligned} x^{2(p-q)-1} &\sim x^{p-3q}, \\ p + q &= 1. \end{aligned} \tag{150}$$

Solving the simultaneous equations 147 and 150 we get

$$p = \frac{1}{3}, \quad q = \frac{2}{3}. \tag{151}$$

So the similarity scaling is

$$\Psi \sim x^{1/3} f(\eta) \;\text{ with }\; \eta \sim \frac{y}{x^{2/3}} \tag{152}$$

[6]For a solid surface, this was specified by the exterior solution $u_e(x)$, Eqn. 112. In contrast here $u_e = 0$.

Similarity Transformation

Using the scalings just established, we now propose the similarity transformation

$$\eta = \frac{1}{3\nu^{1/2}} \frac{y}{x^{2/3}} \quad \text{and} \quad \xi = x \tag{153}$$

with the following expression for the stream function,

$$\Psi = \nu^{1/2} x^{1/3} f(\eta). \tag{154}$$

The change of coordinate system from (x, y) to (ξ, η) leads to:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} - \frac{2}{3} \frac{\eta}{x} \frac{\partial}{\partial \eta}, \tag{155}$$

and

$$\frac{\partial}{\partial y} = \frac{x^{-2/3}}{3\nu^{1/2}} \frac{\partial}{\partial \eta}. \tag{156}$$

The expressions for u and v in the new coordinate system are then

$$u = \frac{\partial \Psi}{\partial y} = \frac{x^{-1/3}}{3} f'(\eta), \tag{157}$$

and

$$v = -\frac{\partial \Psi}{\partial x} = -\frac{\nu^{1/2}}{3} x^{-2/3} \{f - 2\eta f'\}. \tag{158}$$

Substituting these into the momentum equation 135 we get

$$\begin{aligned} \nu \frac{x^{-2/3}}{3\nu^{1/2}} \frac{\partial}{\partial \eta} \left\{ \frac{x^{-2/3}}{3\nu^{1/2}} \frac{\partial}{\partial \eta} \left(\frac{x^{-1/3}}{3} f' \right) \right\} &= \frac{x^{-1/3}}{3} f' \left\{ \frac{\partial}{\partial x} - \frac{2\eta}{3x} \frac{\partial}{\partial \eta} \right\} \left(\frac{x^{-1/3}}{3} f' \right) \\ &- \frac{\nu^{1/2}}{3} x^{-2/3} \{f - 2\eta f'\} \frac{x^{-2/3}}{3\nu^{1/2}} \frac{\partial}{\partial \eta} \left(\frac{x^{-1/3}}{3} f' \right). \end{aligned}$$

Multiplying across by $27x^{5/3}$ and tidying up we get

$$f''' + ff'' + f'^2 = 0, \tag{159}$$

which is the final simplified form of the momentum equation, expressed in terms of the scaled stream function. Its boundary conditions are deduced as follows:

- Zero velocity $u \to 0$ far from the jet gives $f'(\infty) \to 0$.
- Symmetry about the jet's centre: $v = 0$ gives $f(0) = 0$ and $\partial u/\partial y = 0$ gives $f''(0) = 0$.

Solution

Eqn. 159 can be solved analytically. We start by rewriting it as

$$\begin{aligned} f''' + \frac{d}{d\eta}(ff') &= 0, \\ f'' + ff' &= C. \end{aligned} \qquad (160)\ (161)$$

The constant $C = 0$ since $f(0) = f''(0) = 0$. We now make the transformation

$$\chi = \alpha\eta, \quad f(\eta) = 2\alpha F(\chi), \qquad (162)$$

where α is a free constant, to be determined later. Eqn. 161 then becomes

$$\begin{aligned} 2\alpha^3 F'' + 4\alpha^3 FF' &= 0, \\ F'' + 2FF' &= 0, \end{aligned} \qquad (163)$$

with BCs $F(\chi = 0) = 0$ and $F'(\chi = \infty) = 0$. Integrating once more we get

$$F' = 1 - F^2, \qquad (164)$$

in which the integration constant (the first term on the RHS) was chosen equal to 1. We were able to do this without loss of generality, because of the freedom in α. Upon integration, this yields the following solution:

$$\chi = \int_0^F \frac{dF}{1-F^2} = \frac{1}{2}\ln\left(\frac{1+F}{1-F}\right) = \tanh^{-1}(F), \qquad (165)$$

and so

$$F = \tanh(\chi). \qquad (166)$$

Recalling 162 we now have

$$f'(\eta) = 2\alpha^2 F'(\chi) = 2\alpha^2 \text{sech}^2(\chi), \qquad (167)$$

which gives the solution for the horizontal component of velocity:

$$u = \frac{1}{3x^{1/3}} f'(\eta) = \frac{2}{3}\frac{\alpha^2}{x^{1/3}}\text{sech}^2(\alpha\eta). \qquad (168)$$

It remains to calculate α. For an imposed momentum flux M this is given by

$$\begin{aligned} M &= \rho\int_{-\infty}^{+\infty} u^2 dy \\ &= 2\rho\int_0^\infty \left(\frac{4}{9}\frac{\alpha^4}{x^{2/3}}\right)\left(3\nu^{1/2}\frac{x^{2/3}}{\alpha}\right)\text{sech}^4(\chi)d\chi \\ &= \frac{8}{3}\rho\nu^{1/2}\alpha^3\int_0^\infty (1 - \tanh^2\chi)(\tanh\chi)'d\chi \\ &= \frac{8}{3}\rho\nu^{1/2}\alpha^3\left[\tanh\chi - \frac{1}{3}\tanh^3\chi\right]_0^\infty \\ &= \frac{16}{9}\rho\nu^{1/2}\alpha^3, \end{aligned} \qquad (169)$$

giving finally

$$\alpha = \left(\frac{9}{16}\right)^{1/3}\left(\frac{M}{\rho\nu^{1/2}}\right)^{1/3}. \qquad (170)$$

Using this expression for α, together with Eqn. 153 for η, in Eqn. 168 for u, we get finally the horizontal velocity component in the jet:

$$u = \left(\frac{3M^2}{32\rho^2 \nu x}\right)^{\frac{1}{3}} \operatorname{sech}^2 \left[\left(\frac{M}{48\rho\nu^2 x^2}\right)^{\frac{1}{3}} y\right]. \tag{171}$$

An expression for the vertical velocity component can be deduced similarly. The corresponding flow field (u, v) is plotted in Fig. 16. The main point to note is that, as the jet spreads out with increasing downstream distance under the action of viscous diffusion (its thickness scaling as $x^{2/3}$), its typical $x-$velocity scale decreases as $x^{-1/3}$ to ensure that the total $x-$momentum remains constant.

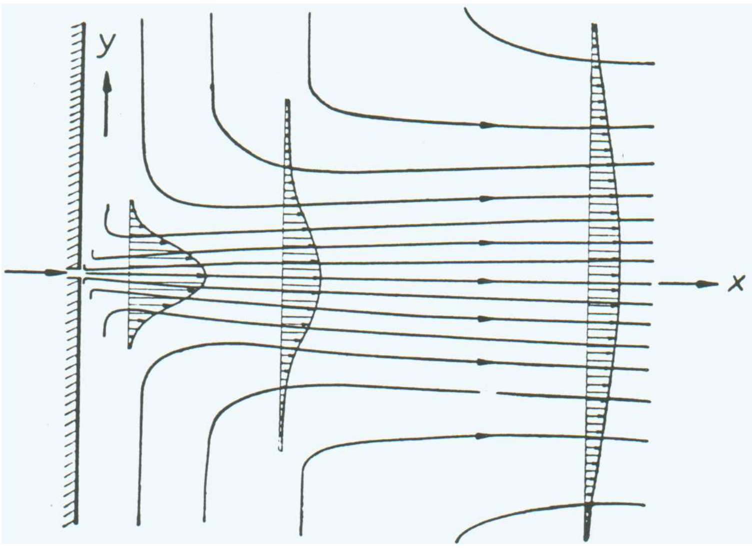

Figure 16: Evolution of the jet velocity profile downstream of the slit. (From Schlichting, 'Boundary Layer Theory', 7th edition, McGraw Hill.)