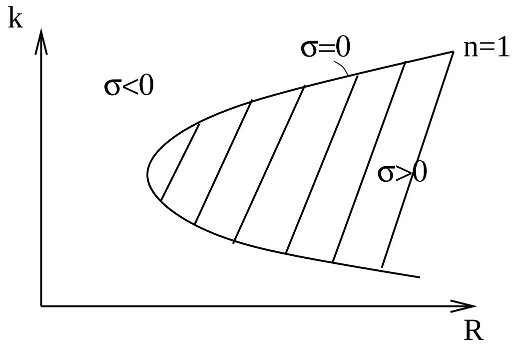
7 Local bifurcation theory

In our linear stability analysis of the Eckhaus equation we saw that a neutral curve is generated, as sketched again below.



As the control parameter R is smoothly varied, the point at which $\sigma = 0$ (at any fixed k) defines the stability threshold or "bifurcation point" at which the base flow switches from being linearly stable to linearly unstable with respect to perturbations of wavevector k. As we will show in Sec. 8, in the regime of linear instability, nonlinear terms in the Eckhaus equation act to restabilise the system somewhat. Because of this, the final nonlinear state is not too "far away" from the original base state. (Recall the diagrams on page 3.) To describe the bifurcation fully, these nonlinear effects must clearly be taken into account.

In this section, we introduce the general theory of bifurcations in the context of some simpler equations. For our purposes, each of these can be viewed as the simplest "standard" equation to capture a given type of bifurcation (saddlenode, transcritical *etc.*). More fundamentally, though, even the most complicated equations can be shown to reduce to one of these standard forms when expanded about a bifurcation point. In Sec. 8 below, for example, we show that the Eckhaus equation exhibits a pitchfork bifurcation at the $R_{\rm cm}, k_{\rm cm}$, described by a simple equation of the form (30).

With these remarks in mind, we now introduce each type of bifurcation in turn.

7.1 The saddlenode bifurcation

Consider the dynamical system defined by

$$\frac{dx}{dt} = a - x^2, \quad \text{for} \quad x, a \quad \text{real}. \tag{16}$$

Here a is a control parameter that can be tuned externally. A steady state solution ($dx/dt = 0$) is simply

$$x = x_{\rm B} = \pm\sqrt{a}. \tag{17}$$

Therefore, for

- $a < 0$ we have no real solutions.
- $a > 0$ we have two real solutions.

We now consider each of the two solutions for $a > 0$, and examine their linear stability in the usual way. First, we add a small perturbation:

$$x = x_{\rm B} + \tilde{x}. \tag{18}$$

Substituting this into the governing equation (16), we get

$$\frac{d\tilde{x}}{dt} = (a - x_{\rm B}^2) - 2x_{\rm B}\tilde{x} - \tilde{x}^2. \tag{19}$$

The term in brackets on the RHS is trivially zero, from (17). At first order in the perturbation, $\tilde{x}$, we therefore have

$$\frac{d\tilde{x}}{dt} = -2x_{\rm B}\tilde{x}, \tag{20}$$

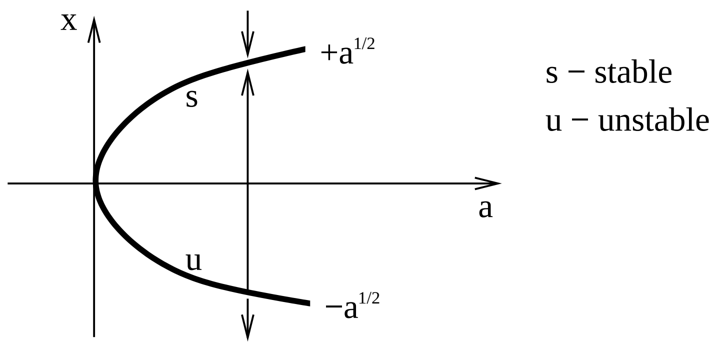
with solution

$$\tilde{x}(t) = A\exp(-2x_{\rm B}t). \tag{21}$$

From this, we see that

- for $x_{\rm B} = +\sqrt{a}$, $|\tilde{x}| \to 0$ as $t \to \infty$ (linear stability);
- for $x_{\rm B} = -\sqrt{a}$, $|\tilde{x}| \to \infty$ as $t \to \infty$ (linear instability).

As sketched in the "bifurcation diagram" below, therefore, the saddlenode bifurcation at $a = 0$ corresponds to the creation of two new solution branches. One of these is linearly stable, the other linearly unstable.



7.2 The transcritical bifurcation

Consider the dynamical system

$$\frac{dx}{dt} = ax - bx^2 \quad \text{for} \quad x, a, b \quad \text{real}. \tag{22}$$

Again, a and b are control parameters. We can find two steady states ($dx/dt = 0$) to this system

- $x = x_{\rm B1} = 0 \qquad \forall \quad a, b.$
- $x = x_{\rm B2} = a/b \quad \forall \quad a, b \quad (b \neq 0).$

We now examine the linear stability of each of these states in turn, following the usual procedure.

Starting with state $x_{\rm B1}$, we add a small perturbation

$$x = x_{\rm B1} + \tilde{x}. \tag{23}$$

This gives

$$\frac{d\tilde{x}}{dt} = a\tilde{x} - b\tilde{x}^2 \tag{24}$$

with the linearised form

$$\frac{d\tilde{x}}{dt} = a\tilde{x}. \tag{25}$$

This has the solution

$$\tilde{x}(t) = A\exp(at). \tag{26}$$

At linear order, therefore, perturbations grow for $a > 0$ and decay for $a < 0$. So

- state $x_{\rm B1} = 0$ is linearly unstable for $a > 0$, and
- state $x_{\rm B1} = 0$ is linearly stable for $a < 0$.

Now consider the linear stability of the second state $x_{\rm B2}$. As usual we write

$$x = x_{\rm B2} + \tilde{x}. \tag{27}$$

Substituting this into the equation of motion (22) and linearising, we get

$$\begin{aligned} \frac{d\tilde{x}}{dt} &= a\tilde{x} - 2bx_{\rm B2}\tilde{x} \\ &= a\tilde{x} - 2b\left(\frac{a}{b}\right)\tilde{x} \\ &= -a\tilde{x}. \end{aligned} \tag{28}$$

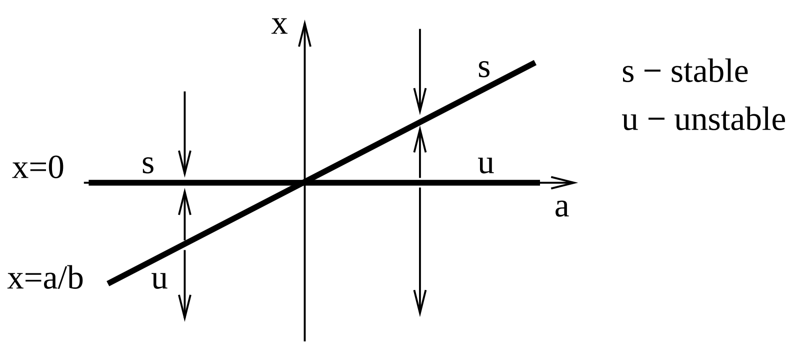
(Do the linearisation as an exercise.) This has the solution

$$\tilde{x}(t) = A\exp(-at), \tag{29}$$

giving exponential growth for $a < 0$ and decay for $a > 0$. Thus we see that

- state $x_{\rm B2} = a/b$ is linearly unstable for $a < 0$, and
- state $x_{\rm B2} = a/b$ is linearly stable for $a > 0$.

These findings are summarised in the following bifurcation diagram. The bifurcation at $a = 0$ corresponds to an exchange of stabilities between the two solution branches.



7.3 The pitchfork bifurcation

Consider the dynamical system defined by

$$\frac{dx}{dt} = ax - bx^3 \quad \text{for} \quad x, a, b \quad \text{real}. \tag{30}$$

As usual, a and b are external control parameters. Steady states, for which $dx/dt = 0$, are as follows:

$$x = x_{\mathrm{B1}} = 0, \tag{31}$$

$$x = x_{\mathrm{B2}} = +\sqrt{a/b} \quad \text{for} \quad a/b > 0, \tag{32}$$

$$x = x_{\mathrm{B3}} = -\sqrt{a/b} \quad \text{for} \quad a/b > 0. \tag{33}$$

So states x_{B2} and x_{B3} only exist for $a > 0$ if $b > 0$; and for $a < 0$ if $b < 0$. In drawing our bifurcation diagrams below, therefore, we will consider the case $b > 0$ separately from the case $b < 0$.

As usual, we now examine the linear stability of each of these steady states in turn. (This can be done for a general b.) First we write

$$x = x_{\mathrm{B1}} + \tilde{x} \tag{34}$$

and find the linearised equation

$$\frac{d\tilde{x}}{dt} = a\tilde{x}, \tag{35}$$

with the solution

$$\tilde{x} = A\exp(at). \tag{36}$$

So we see that

- state $x_{\mathrm{B1}} = 0$ is linearly unstable for $a > 0$, and
- state $x_{\mathrm{B1}} = 0$ is linearly stable for $a < 0$.

The linear stability of states $x = x_{\mathrm{B2}}$ and $x = x_{\mathrm{B3}}$ can be considered together. Setting

$$x = \pm\sqrt{a/b} + \tilde{x} \tag{37}$$

we get, at linear order in $\tilde{x}$, the equation

$$\frac{d\tilde{x}}{dt} = a\tilde{x} - 3bx_{\mathrm{B}}^2\tilde{x} \tag{38}$$

with the solution

$$\tilde{x} = A\exp(st) \tag{39}$$

in which

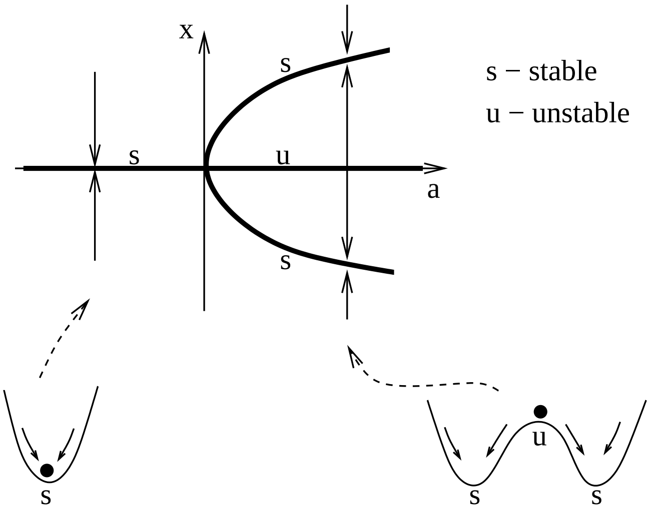
$$s = a - 3bx_{\mathrm{B}}^2 = a - 3b\frac{a}{b} = -2a. \tag{40}$$

Thus we see that

- states x_{B2} and x_{B3} are linearly stable for $a > 0$, and
- states x_{B2} and x_{B3} are linearly unstable for $a < 0$.

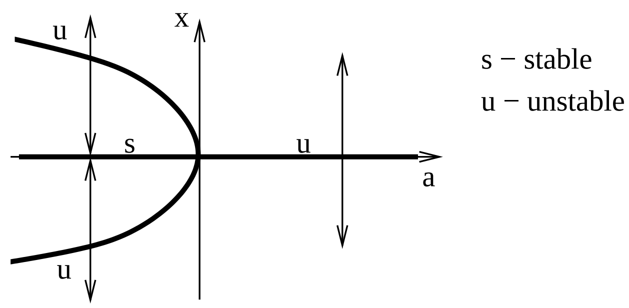
We now collect these results in bifurcation diagrams in the plane $x - a$. As noted above, we will do this separately for $b > 0$ and $b < 0$. From (32) and (33), we recall that the states $x = x_{B2}$ and $x = x_{B3}$ only exist for $a/b > 0$. So when $b > 0$, they only exist for $a > 0$. Given this, and the stability properties deduced above, we have the "supercritical pitchfork" bifurcation diagram sketched below. Nonlinearity has a stabilising influence in this case. In the particle-in-a-well analogy, this corresponds to the bottom right sketch on page 2.

b>0   supercritical pitchfork bifurcation

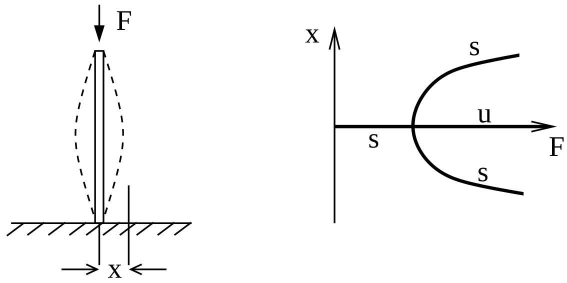


When $b < 0$, states $x = x_{B2}$ and $x = x_{B3}$ only exist for $a < 0$. Given this, and the stability properties deduced above, we have the "subcritical pitchfork" bifurcation diagram sketched below. So nonlinearity has a destabilising influence in this case. In the particle-in-a-well analogy, the left hand part of this plot corresponds to the bottom left sketch on page 2.

b<0   subcritical pitchfork bifurcation



A physical example − Pitchfork bifurcations are common in physical systems that possess an underlying symmetry. This is intuitively obvious, because (30) is invariant under the transformation $x \to -x$. One physical example is the so-called Euler strut. Here we apply an increasing load to a vertical strut, until it finally buckles. Right and left buckling are equivalent: the symmetry $x \to -x$ applies. A detailed analysis of the problem shows that the system does indeed suffer a supercritical pitchfork bifurcation at the point of buckling. We will discuss some other physical examples in Sec. 8 below.



7.4 The Hopf bifurcation

Consider the dynamical system defined by the two equations

$$\frac{dx}{dt} = -y + (a - x^2 - y^2)x$$
$$\frac{dy}{dt} = x + (a - x^2 - y^2)y. \tag{41}$$

for real x, y, a. There is a trivial steady state at $x = y = 0$. To examine its linear stability, we write

$$x = 0 + \tilde{x}, \quad y = 0 + \tilde{y}. \tag{42}$$

Substituting this into the defining equations (41), and linearising, we get

$$\frac{d\tilde{x}}{dt} = -\tilde{y} + a\tilde{x},$$
$$\frac{d\tilde{y}}{dt} = \tilde{x} + a\tilde{y}. \tag{43}$$

The solution of these linearised equations has the form

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \exp(st) + \text{c.c.} \tag{44}$$

Substituting this into (43), we find the eigenvalue s and the eigenvector (α, β) to be determined by the following system of linear equations

$$\alpha s = -\beta + a\alpha$$
$$\beta s = \alpha + a\beta. \tag{45}$$

(Check this as an exercise.) Eliminating α and β, we find the following equation for the eigenvalue s at any a:

$$s^2 - 2as + (a^2 + 1) = 0, \tag{46}$$

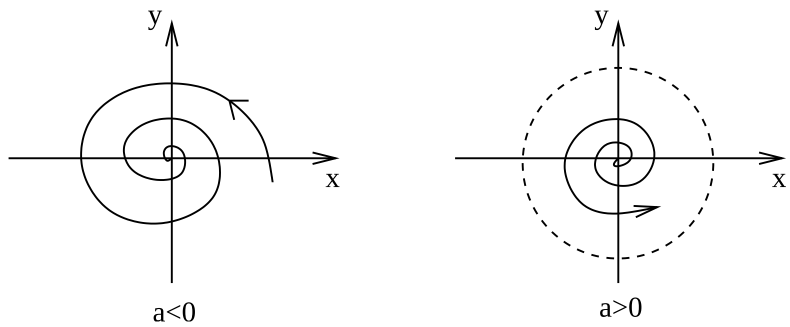
from which it is easy to show that the eigenvalues are

$$s = a \pm i. \tag{47}$$

(In principle, we could substitute these back into (45) to find the corresponding eigenvectors (α, β), but do not pursue this here.) Given (44) and (47), we see that

- if $a > 0$ then $\Re(s) > 0$ and so $|\tilde{x}|, |\tilde{y}| \to \infty$ (linear instability);
- if $a < 0$ then $\Re(s) < 0$ and so $|\tilde{x}|, |\tilde{y}| \to 0$ (linear stability).

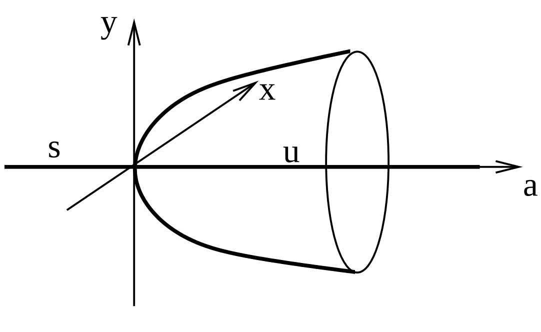
The fact that s is complex confers a new dynamical feature not encountered in the previous examples: that of temporal oscillation. For $a < 0$, for example, the progress of $\tilde{x}$ and $\tilde{y}$ in towards the origin is via a damped oscillation, as sketched in the left hand plot, rather than a straightforward exponential decay.



As in the other bifurcation examples, the loss of stability at $a = 0$ gives rise to a new solution for $a > 0$. In this case, the new solution is periodic:

$$x = \sqrt{a}\cos(t + t_0), \quad y = \sqrt{a}\sin(t + t_0). \tag{48}$$

The system orbits round the "limit cycle" drawn by the dashed line in the right hand sketch above. The bifurcation diagram is then as follows.



Comparing this to the diagrams on page 12, you will notice that it looks a bit like a higher dimensional version of a supercritical pitchfork bifurcation. Indeed, we can again classify Hopf bifurcations as supercritical or subcritical, according to whether the nonlinearity is destabilising or stabilising respectively.

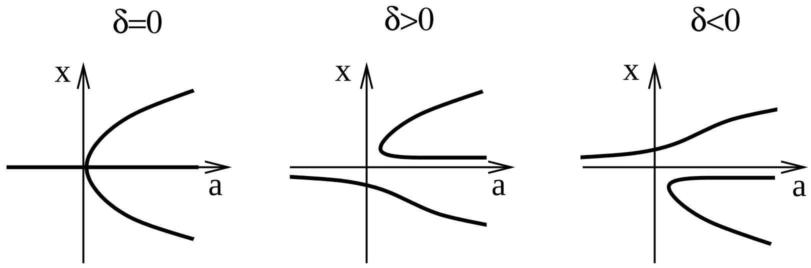
7.5 Imperfection theory / structural stability

As noted earlier, pitchfork bifurcations are common in systems that possess an underlying symmetry: $x \to -x$ in the notation used here. In many real situations, however, the symmetry is only approximate: imperfections lead to a slight difference between left and right (or whatever the relevant opposite generalised displacements are). In this section, we are concerned with what happens when such small imperfections are present.

Consider a slightly imperfect version of (30), in which we choose to set $b = 1$.

$$\frac{dx}{dt} = ax - x^3 - \delta. \tag{49}$$

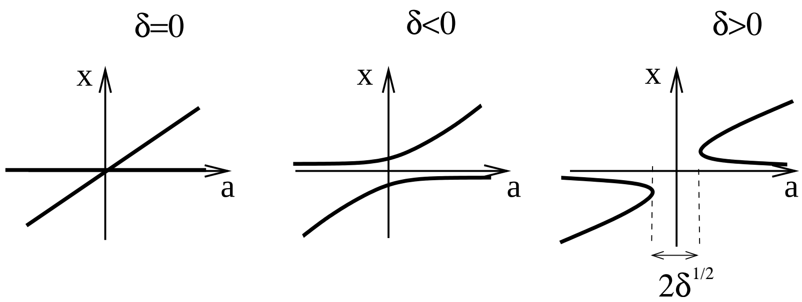
Here δ, which is assumed small, is a measure of the degree of imperfection present. If $\delta = 0$ we have steady states at $x = 0$ and $x = \pm\sqrt{a}$, with a pitchfork bifurcation at $a = 0$, as considered previously. When $\delta \neq 0$, however, we have steady states for $a = x^2 + \delta/x$, and the bifurcation diagram is modified as follows:



Consider now a slightly imperfect version of (22), in which we choose to set $b = 1$:

$$\frac{dx}{dt} = ax - x^2 - \delta. \tag{50}$$

Again, for $\delta = 0$ we have steady states at $x = 0$, $x = a$, and a transcritical bifurcation at $a = 0$. For $\delta \neq 0$, however, we have steady states at $x = \frac{1}{2}\left[a \pm \sqrt{a^2 - 4\delta}\right]$. For small $|\delta|$, the bifurcation diagram is modified as follows. In particular, we note that if $\delta > 0$ then there is no steady solution for $a^2 < 4\delta$.



Both the pitchfork and transcritical bifurcations are said to be structurally unstable, since they suffer a qualitative topological change when the governing equation is perturbed slightly.

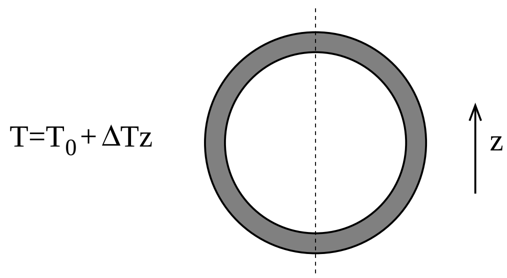
7.6 Bifurcations in the Lorentz equations

In this section, we consider the bifurcations that are exhibited by the Lorentz equations

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y-x),\\ \frac{dy}{dt} &= rx - y - xz,\\ \frac{dz}{dz} &= -bz + xy. \end{aligned} \tag{51}$$

As usual, x, y, z are real dynamical variables; and σ, r, b are control parameters, which we take to be real and positive. Throughout we will assume σ, b to be fixed, and work with r as the single control parameter to be varied.

The Lorentz equations arise in modelling convection in a vertical torus, sketched below. We do not discuss this physical motivation any further here: details can be found in "Physical Fluid Dynamics" by Tritton if you are interested.



In what follows, our aim will be first to find stationary states of the Lorentz equations and then to examine the linear stability of these states. In doing so, we shall demonstrate the existence of a supercritical pitchfork bifurcation and subcritical Hopf bifurcations in the model. Finally, we will briefly discuss the possible scenarios that arise following the loss of stability at a subcritical bifurcation, in which there is no "nearby" nonlinear state to settle to.

7.6.1 Stationary states

- By inspection, we can easily see that there is a trivial stationary state

$$(x_{\mathrm{B1}}, y_{\mathrm{B1}}, z_{\mathrm{B1}}) = (0, 0, 0). \tag{52}$$

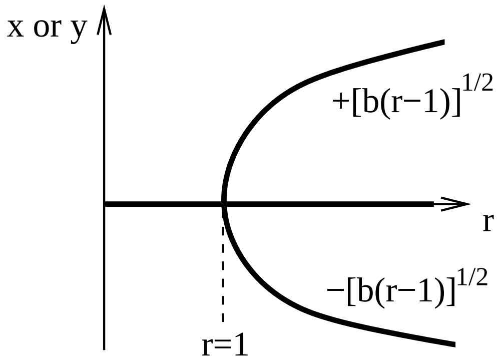
- Another stationary state can be found as follows

$$\begin{aligned}\frac{dx}{dt} &= 0 \quad \text{gives} \quad x = y,\\ \frac{dy}{dt} &= 0 \quad \text{gives} \quad x(r-1) - xz = 0,\\ \frac{dz}{dt} &= 0 \quad \text{gives} \quad -bz + x^2 = 0. \end{aligned} \tag{53}$$

From the second of these we get $z = r - 1$. Putting this into the third, we get $x^2 = b(r-1)$. Combined with the first, $x = y$, we get finally the stationary state

$$(x_{\mathrm{B2}}, y_{\mathrm{B2}}, z_{\mathrm{B2}}) = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1). \tag{54}$$

These stationary states are collected on a bifurcation diagram as follows.



7.6.2 Linear stability

We now examine the linear stability of each of these stationary states. As usual, we set $x = x_{\mathrm{B}} + \tilde{x}$, $y = y_{\mathrm{B}} + \tilde{y}$, $z = z_{\mathrm{B}} + \tilde{z}$ and linearise the equations in $\tilde{x}, \tilde{y}, \tilde{z}$. This gives

$$\begin{aligned} \frac{d\tilde{x}}{dt} &= \sigma(\tilde{y} - \tilde{x}), \\ \frac{d\tilde{y}}{dt} &= r\tilde{x} - \tilde{y} - x_{\mathrm{B}}\tilde{z} - z_{\mathrm{B}}\tilde{x}, \\ \frac{d\tilde{z}}{dt} &= -b\tilde{z} + x_{\mathrm{B}}\tilde{y} + y_{\mathrm{B}}\tilde{x}. \end{aligned} \tag{55}$$

- For the trivial base state $(x_{\mathrm{B1}}, y_{\mathrm{B1}}, z_{\mathrm{B1}}) = (0, 0, 0)$, these reduce to

  $$\frac{d\tilde{x}}{dt} = \sigma(\tilde{y} - \tilde{x}), \quad \frac{d\tilde{y}}{dt} = r\tilde{x} - \tilde{y}, \quad \frac{d\tilde{z}}{dt} = -b\tilde{z}. \tag{56}$$

  The dynamics of $\tilde{z}$ is trivial: the third equation gives simple exponential decay, $\tilde{z} = \gamma \exp(-bt)$ where γ is a constant. The equations for $\tilde{x}$ and $\tilde{y}$ are coupled. We therefore seek a solution of the form $\tilde{x} = \alpha \exp(st)$ and $\tilde{y} = \beta \exp(st)$. In doing so, we obtain

  $$\begin{aligned} \alpha s &= \sigma(\beta - \alpha), \\ \beta s &= r\alpha - \beta. \end{aligned} \tag{57}$$

  This linear eigenvalue problem has a nontrivial solution if

  $$\begin{vmatrix} s + \sigma & -\sigma \\ -r & s + 1 \end{vmatrix} = 0 \tag{58}$$

  and so if

  $$(s + \sigma)(s + 1) - \sigma r = 0. \tag{59}$$

  Solving this quadratic equation for s gives

  $$s = \frac{1}{2}\left\{-(\sigma + 1) \pm \sqrt{(\sigma + 1)^2 - 4\sigma(1 - r)}\right\}. \tag{60}$$

  This gives $\Re s < 0$ (linear stability) for $r < 1$ and $\Re s > 0$ (linear instability) for $r > 1$. The bifurcation at $r = 1$ is a supercritical pitchfork.

- We now analyse the linear stability of the state $(x_{\rm B2}, y_{\rm B2}, z_{\rm B2})$. For r just greater than 1, we expect this to be linearly stable, consistent with the supercritical pitchfork bifurcation that we have just discussed above at $r = 1$. The following analysis will confirm this, but will also reveal a secondary instability in the form of a subcritical Hopf bifurcation at a value $r = r_{\rm crit} > 1$, to be determined.

  Inserting $(x_{\rm B2}, y_{\rm B2}, z_{\rm B2})$ into (55), and seeking solutions to the resulting equation set in the form

$$\tilde{x} = \alpha \exp(st), \quad \tilde{y} = \beta \exp(st), \quad \tilde{z} = \gamma \exp(st), \tag{61}$$

  we find the following polynomial equation for the eigenvalue s

$$s^3 + (\sigma + b + 1)s^2 + b(\sigma + r)s + 2b\sigma(r - 1) = 0. \tag{62}$$

  One can show that the only possibility in this case is a Hopf bifurcation: *i.e.* that the eigenvalue has non-zero imaginary part $\Im s \neq 0$ at the bifurcation point where the real part changes sign, $\Re s = 0$. So we insert a solution in the form $s = i\omega$ for ω real into (62). Taking real and imaginary parts, we then get

$$-\omega^3 + b(\sigma + r)\omega = 0 \tag{63}$$

  and

$$-\omega^2(\sigma + b + 1) + 2b\sigma(r - 1) = 0. \tag{64}$$

  From (64) we get

$$\omega = \pm\sqrt{\frac{2b\sigma(r-1)}{\sigma + b + 1}} \quad \text{for} \quad r > 1. \tag{65}$$

  Combining this with the requirement from (63) that (for $\omega \neq 0$)

$$\omega^2 = b(\sigma + r) \tag{66}$$

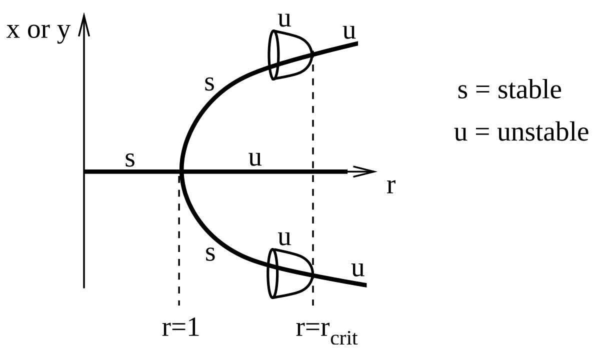
  we get

$$2b\sigma(r - 1) = b(\sigma + r)(\sigma + b + 1), \tag{67}$$

  which can be rearranged to give

$$r_{\rm crit} = \sigma(3 + b + \sigma)/(\sigma - b - 1) \tag{68}$$

  This Hopf bifurcation can be shown to be subcritical. Collecting all the above results together, we get finally the following bifurcation diagram.



7.6.3 Dynamical evolution beyond subcritical bifurcations

In the bifurcation diagram sketched above, we discussed the existence of a subcritical bifurcation at $r = r_{\text{crit}}$. What happens for $r > r_{\text{crit}}$, where stability is lost and there is no “nearby” nonlinear state to go to? In general, several scenarios are possible:

- Evolution to infinity, typically indicating a breakdown of the model.
- Evolution to a non-local fixed point.
- Evolution to a non-local periodic or quasi-periodic state.
- Evolution to a strange attractor, leading to chaotic dynamics.

In the Lorentz equations just discussed, the last of these scenario occurs for $r > r_{\text{crit}}$.