8 The Stuart-Landau equation

8.1 Derivation

In Sec. 6 above, we studied the linear stability properties of the Eckhaus equation

$$\frac{1}{R}\left[\Phi_{\eta\eta}+\Phi_{\xi\xi}\right]-\Phi_{\xi\xi\xi\xi}-\Phi_t=\Phi_\eta\Phi_{\xi\xi}. \tag{69}$$

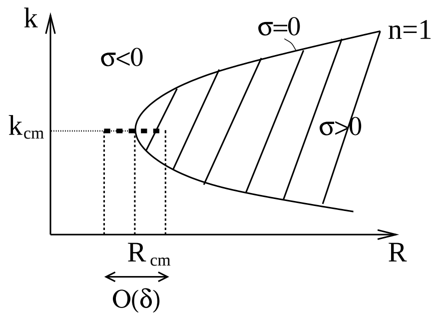
As usual, we expressed Φ as the sum of a base state $\Phi_{\rm B}=\eta$ plus a small deviation,

$$\Phi=\Phi_{\rm B}+\epsilon\cos(k\xi)\sin(n\pi\eta)\exp(\sigma t), \tag{70}$$

and showed the growth rate σ to depend on the control parameter R, on the wavevector k and on the wavenumber n as

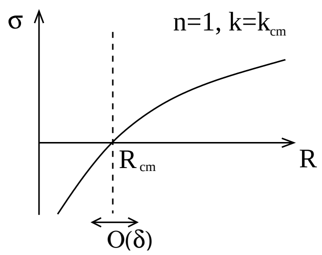
$$\sigma=k^2-k^4-R^{-1}\left[n^2\pi^2+k^2\right]. \tag{71}$$

For the lowest mode $n=1$, this gives the neutral stability curve (on which $\sigma=0$) sketched below. For any fixed value of k, the point at which $\sigma=0$ marks a bifurcation at which the base flow switches from being linearly stable ($\sigma<0$) to linearly unstable ($\sigma>0$) with respect to perturbations of wavevector k.



We now turn to the nonlinear dynamics of the Eckhaus equation. Our aim is to derive a simple nonlinear dynamical equation governing the amplitude of perturbations to the base state in the vicinity of bifurcation, where the nonlinearity remains weak. For simplicity we consider the single, fixed value of the wavevector $k=k_{\rm cm}$ marked in the above sketch, for values of the control parameter $R=R_{\rm cm}+\delta R_1$. Here $R_1=O(1)$ and δ is a small parameter characterising the distance from the bifurcation, as marked by the thick dashed line above. Using this nonlinear equation, we will show that the model exhibits a pitchfork bifurcation as R is tracked through $R_{\rm cm}$ along this thick dashed line at fixed $k=k_{\rm cm}$.

Along this line, the growth rate has the form sketched roughly as follows:



In the vicinity $R = R_{\rm cm} + \delta R_1$ of $R_{\rm cm}$, the behaviour of σ can be examined by setting $n = 1, k = k_{\rm cm}$ in Eqn. 71 and Taylor expanding the resulting expression about $R_{\rm cm}$:

$$\begin{aligned}\sigma &= \sigma(R_{\rm cm}) + \delta R_1 \frac{\partial \sigma}{\partial R}(R = R_{\rm cm}) + \cdots \\ &= \delta R_1 \frac{1}{R_{\rm cm}^2}(\pi^2 + k_{\rm cm}^2) + \cdots, \end{aligned} \tag{72}$$

in which we have set $\sigma(R_{\rm cm}) = 0$ by definition of the bifurcation point. For values of $R = R_{\rm cm} + \delta R_1$ near the bifurcation point, therefore, the growth rate $\sigma = O(\delta)$, which is small. This suggests that we define a slow timescale, suited to tracking the slow evolution of the perturbations:

$$\tau = \delta\, t = O(1) \quad \text{giving} \quad \frac{\partial}{\partial t} \to \delta \frac{\partial}{\partial \tau}. \tag{73}$$

Our aim now is to seek a solution to the Eckhaus equation (69), treating δ as a small parameter. To do so, it helps first to collect together for convenient reference the δ dependencies of the various quantities discussed above:

$$k = k_{\rm cm}, \tag{74}$$

$$R = R_{\rm cm} + \delta R_1, \tag{75}$$

$$\sigma = \delta R_1 \frac{1}{R_{\rm cm}^2}(\pi^2 + k_{\rm cm}^2) + \cdots \tag{76}$$

$$\tau = \delta\, t = O(1) \quad \text{giving} \quad \frac{\partial}{\partial t} \to \delta \frac{\partial}{\partial \tau}. \tag{77}$$

We also assume a solution of the form

$$\Phi = \eta + \left\{ \delta^{1/2}\phi_1 + \delta\phi_2 + \delta^{3/2}\phi_3 + \cdots \right\}, \tag{78}$$

where η is the usual base state and $\{\}$ is the perturbation. The boundary conditions on the ϕ_m are as follows

$$\phi_m = 0 \quad \text{at} \quad \eta = 0, 1 \quad \text{for} \quad m = 1, 2, 3 \cdots \tag{79}$$

Our basic tactic is to substitute (75), (77) and (78) into (69), and consider each successive order $\delta^{1/2}$, δ and $\delta^{3/2}$ in turn. Because these expansions are quite cumbersome, however, it helps to consider the various terms in (69) separately in turn.

On the LHS, we have

$$\begin{aligned}\frac{1}{R}\left[\Phi_{\eta\eta} + \Phi_{\xi\xi}\right] &= \left(\frac{1}{R_{\rm cm}} - \frac{\delta R_1}{R_{\rm cm}^2} + \cdots\right) \\ &\quad \times \left[\delta^{1/2}(\phi_{1\,\eta\eta} + \phi_{1\,\xi\xi}) + \delta(\phi_{2\,\eta\eta} + \phi_{2\,\xi\xi}) + \delta^{3/2}(\phi_{3\,\eta\eta} + \phi_{3\,\xi\xi}) + \cdots\right] \\ &= \delta^{1/2}\frac{1}{R_{\rm cm}}(\phi_{1\,\eta\eta} + \phi_{1\,\xi\xi}) + \delta\frac{1}{R_{\rm cm}}(\phi_{2\,\eta\eta} + \phi_{2\,\xi\xi}) \\ &\quad + \delta^{3/2}\left[\frac{1}{R_{\rm cm}}(\phi_{3\,\eta\eta} + \phi_{3\,\xi\xi}) - \frac{R_1}{R_{\rm cm}^2}(\phi_{1\,\eta\eta} + \phi_{1\,\xi\xi})\right] + \cdots, \end{aligned} \tag{80}$$

together with the fourth derivative

$$-\Phi_{\xi\xi\xi\xi} = -\delta^{1/2}\phi_{1\,\xi\xi\xi\xi} - \delta\phi_{2\,\xi\xi\xi\xi} - \delta^{3/2}\phi_{3\,\xi\xi\xi\xi} + \cdots, \tag{81}$$

and the time derivative

$$-\Phi_t = -\delta^{3/2}\phi_{1\,\tau} + \cdots \tag{82}$$

Note that this comes in only at $O(\delta^{3/2})$, because of the prefactor δ in (77).

On the RHS of (69) we have the nonlinear term

$$\begin{aligned}\Phi_\eta \Phi_{\xi\xi} &= (1+\delta^{1/2}\phi_{1\,\eta}+\delta\phi_{2\,\eta}+\delta^{3/2}\phi_{3\,\eta}+\cdots)\times(\delta^{1/2}\phi_{1\,\xi\xi}+\delta\phi_{2\,\xi\xi}+\delta^{3/2}\phi_{3\,\xi\xi}+\cdots)\\ &= \delta^{1/2}\phi_{1\,\xi\xi}+\delta(\phi_{2\,\xi\xi}+\phi_{1\,\eta}\phi_{1\,\xi\xi})+\delta^{3/2}(\phi_{3\,\xi\xi}+\phi_{1\,\eta}\phi_{2\,\xi\xi}+\phi_{2\,\eta}\phi_{1\,\xi\xi})+\cdots \quad (83)\end{aligned}$$

We can now write equation (69) at each successive order $\delta^{1/2}, \delta$ and $\delta^{3/2}$ simply by collecting the relevant terms from (80) to (83).

- At $O(\delta^{1/2})$ we have

$$\frac{1}{R_{\rm cm}}(\phi_{1\,\eta\eta}+\phi_{1\,\xi\xi})-\phi_{1\,\xi\xi\xi\xi}=\phi_{1\,\xi\xi}. \quad (84)$$

This has the solution

$$\phi_1 = A(\tau)\sin(\pi\eta)\exp(ik_{\rm cm}\xi)+\text{c.c.} \quad (85)$$

(As usual, c.c. denotes complex conjugate.) At this order, the amplitude $A(\tau)$ is undetermined.

- At $O(\delta)$ we have

$$\frac{1}{R_{\rm cm}}(\phi_{2\,\eta\eta}+\phi_{2\,\xi\xi})-\phi_{2\,\xi\xi\xi\xi}=\phi_{2\,\xi\xi}+\phi_{1\,\eta}\phi_{1\,\xi\xi}. \quad (86)$$

Assembling on the LHS all terms in ϕ_2, and substituting ϕ_1 from (85) on the RHS, we get

$$\begin{aligned}\frac{1}{R_{\rm cm}}(\phi_{2\,\eta\eta}+\phi_{2\,\xi\xi})-\phi_{2\,\xi\xi\xi\xi}-\phi_{2\,\xi\xi} &= [\pi A(\tau)\cos(\pi\eta)\exp(ik_{\rm cm}\xi)+\text{c.c.}]\\ &\quad\times\left[-k_{\rm cm}^2A(\tau)\sin(\pi\eta)\exp(ik_{\rm cm}\xi)+\text{c.c}\right]\\ &= -\frac{\pi k_{\rm cm}^2}{2}\sin(2\pi\eta)\left[A(\tau)\exp(ik_{\rm cm}\xi)+\text{c.c}\right]^2.\end{aligned}$$

From this we find

$$\phi_2=\lambda_1A^2\sin(2\pi\eta)\exp(2ik_{\rm cm}\xi)+\text{c.c.}+\lambda_2A\bar{A}\sin(2\pi\eta), \quad (87)$$

in which $\bar{A}(\tau)$ denotes the complex conjugate of $A(\tau)$. λ_1 and λ_2 are constants, which we do not specify here.

- At $O(\delta^{3/2})$ we have

$$\frac{1}{R_{\rm cm}}(\phi_{3\,\eta\eta}+\phi_{3\,\xi\xi})-\frac{R_1}{R_{\rm cm}^2}(\phi_{1\,\eta\eta}+\phi_{1\,\xi\xi})-\phi_{3\,\xi\xi\xi\xi}-\phi_{1\,\tau}=\phi_{3\,\xi\xi}+\phi_{1\,\eta}\phi_{2\,\xi\xi}+\phi_{2\,\eta}\phi_{1\,\xi\xi}.$$

Assembling on the LHS all terms in ϕ_3, and on the RHS all terms in ϕ_1, ϕ_2, we get

$$\frac{1}{R_{\rm cm}}(\phi_{3\,\eta\eta}+\phi_{3\,\xi\xi})-\phi_{3\,\xi\xi\xi\xi}-\phi_{3\,\xi\xi}=\phi_{1\,\tau}+\phi_{1\,\eta}\phi_{2\,\xi\xi}+\phi_{2\,\eta}\phi_{1\,\xi\xi}+\frac{R_1}{R_{\rm cm}^2}(\phi_{1\,\eta\eta}+\phi_{1\,\xi\xi}). \quad (88)$$

The basic strategy now is to substitute (85) and (87) for ϕ_1 and ϕ_2 into the RHS of this equation. The result is unpleasantly messy, containing very many terms. For reasons discussed below, the important ones for our purposes are those in $\sin(\pi\eta)$. Accordingly, we focus only on these.

Examining each term on the RHS for dependence on $\sin(\pi\eta)$, we get:

– RHS, term 1

$$\phi_{1\,\tau} = A_\tau \sin(\pi\eta)\exp(ik_{\rm cm}\xi) + \text{c.c.} \tag{89}$$

– RHS, term 2

$$\begin{aligned}\phi_{1\,\eta}\phi_{2\,\xi\xi} &= [\pi A\cos(\pi\eta)\exp(ik_{\rm cm}\xi) + \text{c.c.}] \\ &\quad \times \left[\lambda_1 A^2 \sin(2\pi\eta).(2ik_{\rm cm})^2 \exp(2ik_{\rm cm}\xi) + \text{c.c}\right] \\ &= c\bar{A}A^2 \sin(\pi\eta)\exp(ik_{\rm cm}\xi) + \text{other terms.}\end{aligned} \tag{90}$$

Here c is a constant, which we do not specify.

– RHS, term 3. This gives the same structure as term 2:

$$\phi_{1\,\xi\xi}\phi_{2\,\eta} = d\bar{A}A^2 \sin(\pi\eta)\exp(ik_{\rm cm}\xi) + \text{other terms.} \tag{91}$$

Again, d is a constant, which we do not specify.

– RHS, term 4

$$\frac{R_1}{R_{\rm cm}^2}(\phi_{1\,\eta\eta} + \phi_{1\,\xi\xi}) = \frac{R_1}{R_{\rm cm}^2} A(-\pi^2 - k_{\rm cm}^2)\sin(\pi\eta)\exp(ik_{\rm cm}\xi) + \text{c.c.} \tag{92}$$

The reason for having focused on the terms in $\sin(\pi\eta)$ is as follows. Recalling that the solution of an equation of the form (88) comprises the sum of a homogeneous solution and a particular integral, we note that the terms in $\sin(\pi\eta)$ on the RHS of (88) "resonate" with terms in the homogeneous solution. (This concept should be familiar from previous courses on differential equations. For the rest of this section, we will use it without further discussion to avoid interrupting our main thread. We will return to revise it in more detail in Sec. 8.2 below.) They therefore lead to a particular integral of the form

$$\eta\cos(\pi\eta) \tag{93}$$

in the full solution for ϕ_3. This clearly cannot satisfy the boundary condition (79), and must therefore vanish identically. Accordingly, the prefactors to the $\sin(\pi\eta)$ terms in (89) to (92) must add to give zero. This "solvability condition" gives finally our main result – the Stuart-Landau equation, governing the weakly nonlinear dynamics of the amplitude of perturbations in the vicinity of bifurcation:

$$\frac{dA}{d\tau} = \sigma_c A - \beta A|A|^2. \tag{94}$$

An equation of this basic form crops up quite generically when examining weakly nonlinear dynamics in the vicinity of bifurcation points. For the example of the Eckhaus equation used here, the constant

$$\sigma_c = \frac{R_1}{R_{\rm cm}^2}(\pi^2 + k_{\rm cm}^2), \tag{95}$$

which is real. The constant β is also real (for the Eckhaus equation), though we do not specify it. Note that in the limit $|A| \to 0$, Eqn. 94 reduces to $A_\tau = \sigma_c A$, reproducing the result of the linear analysis, as required.

8.2 Aside: solvability conditions

In the derivation of the Stuart-Landau (SL) equation, we expanded the Eckhaus equation in successive orders $\delta^{1/2}$, δ and $\delta^{3/2}$. At $O(\delta^{3/2})$ we found

$$\frac{1}{R_{\rm cm}}(\phi_{3\,\eta\eta} + \phi_{3\,\xi\xi}) - \phi_{3\,\xi\xi\xi\xi} - \phi_{3,\xi\xi} = \tilde{F}(A)\sin(\pi\eta)\exp(ik_{\rm cm}\xi) + \text{other terms}, \tag{96}$$

in which

$$\tilde{F}(A) = A_\tau - \sigma_c A + \beta|A|^2 A. \tag{97}$$

In this section, we will discuss in more detail the solvability condition that led us to set $\tilde{F}(A) = 0$, giving the SL equation (94).

We start by setting $\phi_3 = f(\eta)\exp(ik_{\rm cm}\xi)$ in (96). In conjunction with $R_{\rm cm} = (\pi^2 + k_{\rm cm}^2)/(k_{\rm cm}^2 - k_{\rm cm}^4)$, obtained in the linear analysis of Sec. 6, this gives

$$f'' + \pi^2 f = F(A)\sin(\pi\eta) \tag{98}$$

in which

$$F(A) = R_{\rm cm}\tilde{F}(A). \tag{99}$$

Recalling (79), we see that f is subject to the boundary conditions

$$f(0) = 0 \quad \text{and} \quad f(1) = 0. \tag{100}$$

Eqn. 98 has the form of a linear second order ordinary differential equation. Its solution comprises a homogeneous solution $f_{\rm h}$ plus a particular integral $f_{\rm p}$:

$$f = f_{\rm h} + f_{\rm p}. \tag{101}$$

The homogeneous solution

$$f_{\rm h} = \hat{A}\cos(\pi\eta) + \hat{B}\sin(\pi\eta) \tag{102}$$

is the standard solution of the homogeneous equation that is formed by setting the RHS equal to zero in (98). The particular integral has the form

$$f_{\rm p} = \hat{C}\eta\cos(\pi\eta), \tag{103}$$

in which the constant

$$\hat{C} = -\frac{F(A)}{2\pi} \tag{104}$$

is found by substituting (103) into (98). The general solution of (98) is therefore

$$f = \hat{A}\cos(\pi\eta) + \hat{B}\sin(\pi\eta) - \frac{F(A)}{2\pi}\eta\cos(\pi\eta). \tag{105}$$

Applying the boundary condition $f(0) = 0$, we find $\hat{A} = 0$. Applying $f(1) = 0$, we find

$$F(A) = 0. \tag{106}$$

This is the solvability condition. In conjunction with (97) and (99), it gives finally the Stuart-Landau equation (94).

Now let's examine the solvability condition in the context of the more general equation

$$f'' + \pi^2 f = R(\eta), \tag{107}$$

subject to the usual boundary conditions

$$f(0) = 0 \quad \text{and} \quad f(1) = 0. \tag{108}$$

Multiplying (107) across by some function $g(\eta)$ and integrating over the domain, we get

$$\int_0^1 g[f'' + \pi^2 f]d\eta = \int_0^1 g(\eta)R(\eta)d\eta. \tag{109}$$

Integrating by parts twice on the LHS we get

$$[gf' - g'f]_0^1 + \int_0^1 f[g'' + \pi^2 g]d\eta = \int_0^1 g(\eta)R(\eta)d\eta. \tag{110}$$

We now choose $g(\eta)$ so that

$$g'' + \pi^2 g = 0, \tag{111}$$

with boundary conditions

$$g(0) = 0 \quad \text{and} \quad g(1) = 0. \tag{112}$$

(111) and (112) define the so-called "adjoint" problem for g. Together they ensure that the LHS of (110) is zero, so that

$$0 = \int_0^1 g(\eta)R(\eta)d\eta. \tag{113}$$

This is the solvability condition, for a general $R(\eta)$. It requires the solution $g(\eta) = \tilde{B}\sin(\pi\eta)$ of the adjoint problem to be orthogonal to $R(\eta)$ on the domain.

For the special case of $R(\eta) = F(A)\sin(\pi\eta)$ discussed above, (113) gives

$$0 = \tilde{B}F(A)\int_0^1 \sin^2(\pi\eta)dy, \tag{114}$$

which requires $F(A) = 0$ as before. For the different special case of $R(\eta) = F(A)\cos(\pi\eta)$, (113) gives instead

$$0 = \tilde{B}F(A)\int_0^1 \sin(\pi\eta)\cos(\pi\eta)dy. \tag{115}$$

Here we find a problem, because

$$\int_0^1 \sin(\pi\eta)\cos(\pi\eta)dy = 0, \tag{116}$$

i.e. $\sin(\pi\eta)$ and $\cos(\pi\eta)$ are already orthogonal on the interval $[0, 1]$, and (115) therefore does not provide a condition for $F(A)$.

8.3 Predictions of the Stuart-Landau equation

In Sec. 8.1 above, we derived the Stuart-Landau (SL) equation

$$\frac{dA}{d\tau} = \sigma_c A - \beta A|A|^2, \tag{117}$$

which describes weakly nonlinear dynamics in the vicinity of a bifurcation point. Although our derivation was performed in the context of the Eckhaus equation, the SL equation actually emerges quite generically in systems that are close to bifurcation. In many physical models, the constants σ_c and β turn out to be complex. For the simpler case of the Eckhaus equation, recall that they were real.

In this section, we will consider the dynamical behaviour predicted by (117). In order to do so, we will write $A(\tau) = \rho(\tau)e^{i\theta(\tau)}$, and derive corresponding equations of motion for the amplitude ρ and phase θ. We will then analyse the predictions of these, separately for the case of σ_c, β real and complex.

Substituting $A(\tau) = \rho(\tau)e^{i\theta(\tau)}$ into (117), then, we find

$$\dot{\rho}e^{i\theta(\tau)} + \rho i\dot{\theta}e^{i\theta(\tau)} = \sigma_c \rho e^{i\theta(\tau)} - \beta\rho^3 e^{i\theta(\tau)}. \tag{118}$$

Cancelling across the factor $e^{i\theta(\tau)}$, and taking real and imaginary parts of the resulting equation, we get

$$\dot{\rho} = \sigma_{\rm cr}\rho - \beta_{\rm r}\rho^3, \tag{119}$$

and

$$\dot{\theta} = \sigma_{\rm ci} - \beta_{\rm i}\rho^2, \quad \text{for} \quad \rho \neq 0, \tag{120}$$

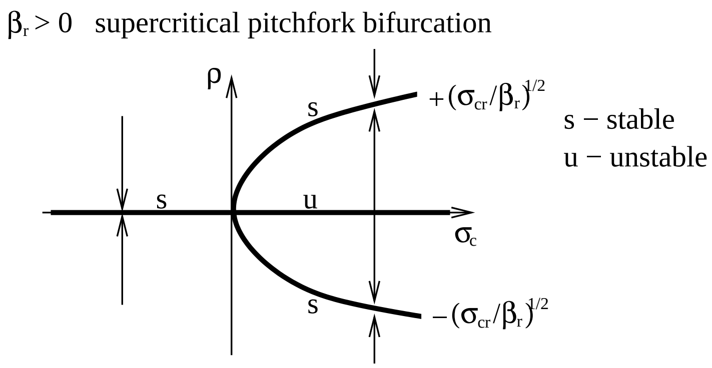
(We have used the usual notation $\beta = \beta_{\rm r} + i\beta_{\rm i}$, and likewise for σ_c.)

8.3.1 σ_c, β real

For σ_c and β real, (119) gives

$$\dot{\rho} = \sigma_{\rm cr}\rho - \beta_{\rm r}\rho^3 \quad \text{and} \quad \dot{\theta} = 0. \tag{121}$$

So the dynamics of the phase $\dot{\theta} = 0$ is trivial. The equation of motion for the amplitude $\dot{\rho} = \cdots$ is of the form of (30), which is the normal form for a pitchfork bifurcation. Thus we have a supercritical pitchfork bifurcation if $\beta_r > 0$ (sketched below) and a subcritical pitchfork bifurcation if $\beta_r < 0$ (recall the lower sketch of page 12) .



8.3.2 Complex: $\sigma_{\rm ci} \neq 0$

The equation of motion for the amplitude is as before

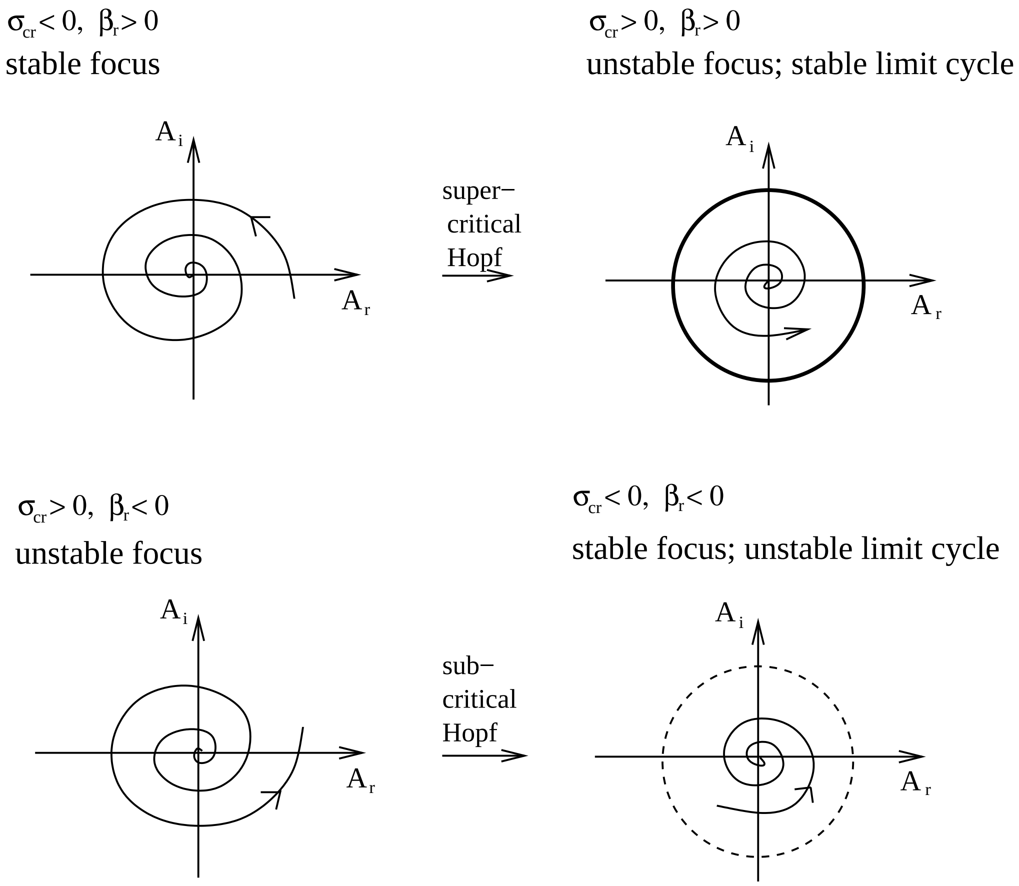
$$\dot{\rho} = \sigma_{\rm cr}\rho - \beta_{\rm r}\rho^3, \tag{122}$$

but now the phase also evolves in time according to

$$\dot{\theta} = \sigma_{\rm ci} + \cdots \tag{123}$$

in which $\cdots$ represents the additional term $-\beta_{\rm i}\rho^2$ that is present if $\beta_{\rm i} \neq 0$. Recalling Q1 of problem sheet 4, therefore, we anticipate Hopf bifurcations in this case.

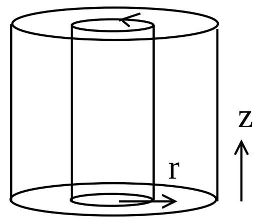
In the Argand plane, ρ gives the distance from the origin, and θ the phase angle. We can thus distinguish the following four scenarios, according to the signs of $\sigma_{\rm cr}$ and $\beta_{\rm r}$. For $\beta_r > 0$, we indeed have a supercritical Hopf bifurcation as σ_{cr} changes sign (recall the sketches on page 14). For $\beta_r < 0$, we have a subcritical Hopf bifurcation.



The sense of rotation (clockwise/anti-clockwise) depends on the sign of σ_{ci}, which has been assumed positive in the above sketches. Switching the sign of σ_{ci} in any of these would not change whether the system spirals into or out of the origin (which is determined by σ_{cr}). It would change the sense of the spiral from anti-clockwise to clockwise.

8.4 Physical examples

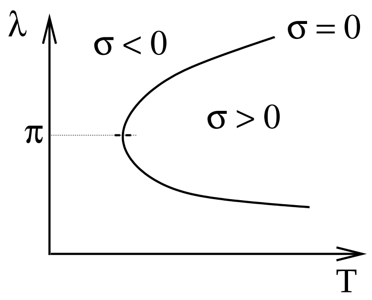
8.4.1 Taylor-Couette instability



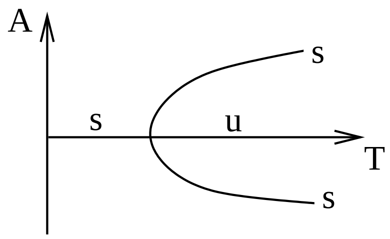
control parameter: Taylor number T

perturbation: A f(r) cos(λz) exp(σt) + ...

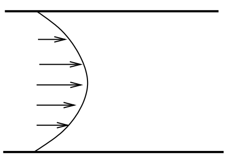
linear stability



supercritical pitchfork
at fixed $\lambda = \pi$



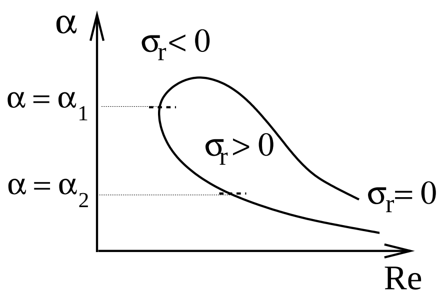
8.4.2 Shear flow instability (planar Poiseuille)



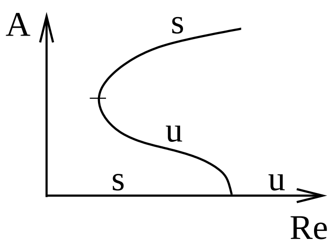
control parameter: Reynolds number Re

perturbation: A f(y) cos(αx) exp(σt) + ...

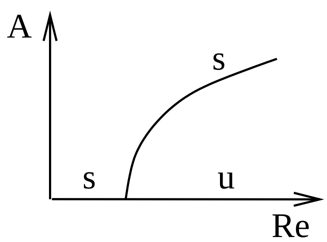
linear stability



subcritical Hopf
at fixed $\alpha = \alpha_1$



supercritical Hopf
at fixed $\alpha = \alpha_2$



$\sigma_i \neq 0$ so Hopf bifurcation