9 The Ginzburg-Landau equation

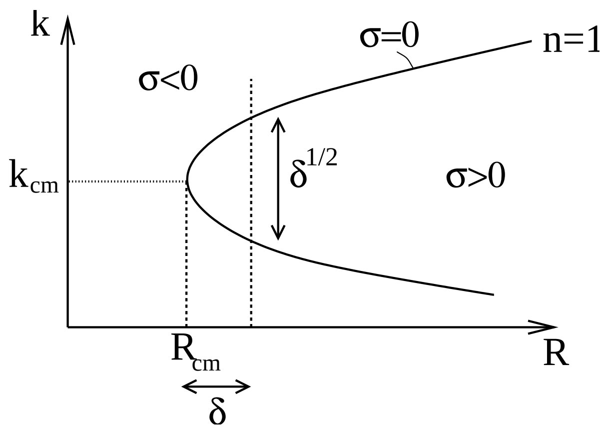
In Sec. 8, we derived the Stuart-Landau equation (94) for the weakly nonlinear dynamics of the amplitude $A(\tau)$ in the vicinity of bifurcation. In the context of the Eckhaus equation, we expanded $R = R_{\rm cm} + \delta R_1$ about the minimum critical value of the control parameter R for small values of δ. For simplicity we considered only the single fixed wavevector $k = k_{\rm cm}$, which is the first to go unstable as R is tracked through $R_{\rm cm}$. Thus we considered perturbations of the form

$$\tilde{\phi} = \delta^{1/2}\phi_1 + \delta\phi_2 + \delta^{3/2}\phi_3 \cdots, \tag{124}$$

in which

$$\phi_1 = A(\tau)\sin(\pi\eta)\exp(ik_{\rm cm}\xi) + \text{c.c.} \tag{125}$$

As seen from the below sketch, however, for any $R = R_{\rm cm} + \delta R_1$ in the unstable regime there is actually a *band* of width $O(\delta^{1/2})$ of unstable wavevectors. In this section, therefore, we relax the assumption of fixed $k = k_{\rm cm}$ and consider the weakly nonlinear dynamics of this entire band. Doing so will lead to the Ginzburg-Landau equation (131). This has a very similar structure to the Stuart-Landau equation, containing only the single additional term μA_{XX} to allow for slow spatial variation of $A = A(X, \tau)$ that arises on the long length scale $X = \delta^{1/2}\xi$ once the band of wavevectors is included.



With the above discussion in mind, we consider a spatial dependence of the form

$$\phi_1 \sim e^{ik\xi} = e^{i(k_{\rm cm}+\delta^{1/2}\tilde{k})\xi} = B(X)e^{ik_{\rm cm}\xi} \tag{126}$$

in which $\tilde{k} = O(1)$ and

$$X = \delta^{1/2}\xi. \tag{127}$$

So as well considering evolution on the slow timescale τ, we now also allow variations on the slow spatial scale X, considering a perturbation of the form

$$\phi_1 = A(X, \tau)\sin(\pi\eta)e^{ik_{\rm cm}\xi} + \text{c.c.} \tag{128}$$

To derive the Ginzburg-Landau equation, we perform an expansion that is closely analogous to the one in our derivation of the Stuart-Landau equation in Sec. 8.1, but now with

$$\frac{\partial}{\partial \xi} \to \frac{\partial}{\partial \xi} + \delta^{1/2}\frac{\partial}{\partial X}. \tag{129}$$

Accordingly, at $O(\delta^{3/2})$ we obtain additional terms of the form

$$\frac{\partial^2 \phi_1}{\partial X^2} \tag{130}$$

on the RHS of the Eckhaus equation. Accounting for these, the amplitude equation becomes

$$A_\tau = \sigma_c A - \beta A|A|^2 + \mu A_{XX}. \tag{131}$$

This is the Ginzburg-Landau (GL) equation. As noted above, it has a very similar structure to the Stuart-Landau equation (94), with the additional term μA_{XX} now allowing for the new dependence of $A = A(X, \tau)$ on the slow spatial scale X.

When derived in the context of the Eckhaus equation, the constants σ_c, β, μ in the GL equation are real. GL equations have also been derived for weakly nonlinear dynamics in the vicinity of bifurcation for Bénard convection, Taylor vortices and Poiseuille flow. In general, the constants σ_c, β, μ can be complex.

9.1 Solution of the Ginzburg-Landau equation

We now consider solutions of the Ginzburg-Landau equation

$$A_\tau = \sigma_c A - \beta A|A|^2 + \mu A_{XX} \tag{132}$$

for the case of real σ_c, β, μ. Our aim will be first to seek a stationary solution in the form $A_e \exp(i\tilde{k}X)$, and then to study the linear stability of this solution.

9.1.1 Stationary solution

Consider a stationary solution in the form

$$A = A_e e^{i\tilde{k}X}, \tag{133}$$

in which we set A_e real WLOG. Substituting this into (132), we get

$$0 = \sigma_c A_e e^{i\tilde{k}X} - \beta A_e^3 e^{i\tilde{k}X} - \mu\tilde{k}^2 A_e e^{i\tilde{k}X}, \tag{134}$$

and so

$$\beta A_e^2 = \sigma_c - \mu\tilde{k}^2. \tag{135}$$

This is essentially the same solution found at the bottom of page 26, but with the additional term $\mu\tilde{k}^2$ arising from the new spatial dependence.

9.1.2 Linear stability

We now consider the linear stability of this stationary state. As usual, we write

$$A = A_e e^{i\tilde{k}X} + B(X,\tau) \tag{136}$$

for $|B| \ll 1$, and linearise the dynamical equation of motion (132). Doing so gives

$$B_\tau(X,\tau) = \sigma_c B - \beta\left(2A_e^2 B + A_e^2 \bar{B} e^{2i\tilde{k}X}\right) + \mu B_{XX}. \tag{137}$$

As usual, $\bar{B}$ denotes the complex conjugate of B. In obtaining (137), we expanded the nonlinear term

$$A|A|^2 = (A_e e^{i\tilde{k}X} + B)(A_e e^{i\tilde{k}X} + B)(A_e e^{-i\tilde{k}X} + \bar{B}), \tag{138}$$

and extracted from this the terms $O(|B|)$ as follows:

$$A_e e^{i\tilde{k}X}(BA_e e^{-i\tilde{k}X} + \bar{B}A_e e^{i\tilde{k}X}) + BA_e^2. \tag{139}$$

These reorganise to give the term in brackets in (137).

We now seek a solution to (137) in the form

$$B(X,\tau) = a(\tau)e^{i\tilde{k}_1 X} + b(\tau)e^{i\tilde{k}_2 X}, \quad \text{with} \quad 2\tilde{k} = \tilde{k}_1 + \tilde{k}_2. \tag{140}$$

Substituting this into (137) and collecting together terms in $\exp(i\tilde{k}_1 X)$ and in $\exp(i\tilde{k}_2 X)$ gives respectively

$$a_\tau = \sigma_c a - 2\beta A_e^2 a - \beta A_e^2 \bar{b} - \mu \tilde{k}_1^2 a, \tag{141}$$

and

$$b_\tau = \sigma_c b - 2\beta A_e^2 b - \beta A_e^2 \bar{a} - \mu \tilde{k}_2^2 b. \tag{142}$$

Substituting βA_e^2 from (135) into (141) gives

$$a_\tau = \sigma_c a - 2[\sigma_c - \mu\tilde{k}^2]a - [\sigma_c - \mu\tilde{k}^2]\bar{b} - \mu\tilde{k}_1^2 a, \tag{143}$$

which can be written in a more compact form

$$a_\tau = \sigma_1 a - 2\sigma_0 a - \sigma_0 \bar{b}, \tag{144}$$

with σ_0, σ_1 defined in (147) below. Substituting βA_e^2 from (135) into (142) gives

$$b_\tau = \sigma_c b - 2[\sigma_c - \mu\tilde{k}^2]b - [\sigma_c - \mu\tilde{k}^2]\bar{a} - \mu\tilde{k}_2^2 b. \tag{145}$$

Taking the complex conjugate of this, and writing in a more compact form, we get

$$\bar{b}_\tau = \sigma_2 \bar{b} - 2\sigma_0 \bar{b} - \sigma_0 a. \tag{146}$$

In the compact forms (144) and (146), we have set

$$\sigma_0 = \sigma_c - \mu\tilde{k}^2, \;\; \sigma_1 = \sigma_c - \mu\tilde{k}_1^2, \quad \text{and} \quad \sigma_2 = \sigma_c - \mu\tilde{k}_2^2. \tag{147}$$

We now seek a solution to (144) and (146) in the form $a(\tau) = \alpha_1 \exp(s\tau)$ and $\bar{b}(\tau) = \bar{\alpha}_2 \exp(s\tau)$. Substituting this into (144) and (146) gives

$$\begin{aligned} s\alpha_1 &= \sigma_1\alpha_1 - 2\sigma_0\alpha_1 - \sigma_0\bar{\alpha}_2, \\ s\bar{\alpha}_2 &= \sigma_2\bar{\alpha}_2 - 2\sigma_0\bar{\alpha}_2 - \sigma_0\alpha_1. \end{aligned} \tag{148}$$

This has a nontrivial solution if

$$\begin{vmatrix} s-\sigma_1+2\sigma_0 & \sigma_0 \\ \sigma_0 & s-\sigma_2+2\sigma_0 \end{vmatrix} = 0. \tag{149}$$

Expanding this determinant gives a quadratic equation in s. In this, it can be shown that $\Re s > 0$ if

$$\tilde{k}^2 > \gamma > 0 \tag{150}$$

for some real constant γ, signifying linear instability of our original stationary solution

$$A_e e^{i\tilde{k}\delta^{1/2}\xi} \sin(\pi\eta) e^{ik_{\text{cm}}\xi} + \text{c.c.} \tag{151}$$

So this solution is unstable if its wavevector $k = k_{\text{cm}} + \delta^{1/2}\tilde{k}$ deviates from k_{cm} by more than the critical amount $\gamma\delta^{1/2}$, as indicated by the shaded area in the below sketch.