Hydrodynamic stability theory

Answers to problem sheet 1. Linear stability.

Q1. Linearising about the trivial basic state, we write

$$u = 0 + \delta\tilde{u} + \cdots. \tag{1}$$

For small δ we have $\sin(u) \approx \delta\tilde{u} + O(\delta^3)$, so at $O(\delta)$ we get the linearised equation

$$\partial_t \tilde{u} - \tilde{u} = \frac{1}{R}\partial_y^2 \tilde{u}. \tag{2}$$

We now do a normal mode analysis, setting $\tilde{u} = \bar{u}(y)\exp(st)$. Substituting this into the linearised equation gives

$$\begin{aligned}(s-1)\bar{u} &= \frac{1}{R}\bar{u}'',\\ \bar{u}'' + R(1-s)\bar{u} &= 0.\end{aligned}$$

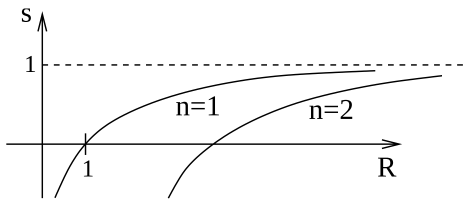
This has a solution of the form

$$\bar{u} = \alpha\cos(\lambda y) + \beta\sin(\lambda y) \quad \text{with} \quad \lambda^2 = R(1-s), \tag{3}$$

in which α and β are constants. The boundary conditions $\bar{u}(0) = \bar{u}(\pi) = 0$ impose $\alpha = 0$ and $\lambda = n$ for $n = 1, 2, \cdots$. Thus we have

$$R(1-s) = n^2 \quad \text{and so} \quad s = 1 - \frac{n^2}{R}. \tag{4}$$

From the sketch, we see that the base state is linearly stable for $R < 1$.



Q2. This was discussed in the notes. Starting with Eqn. 53,

$$\left[D^2 - a^2 - \frac{s}{\kappa}\right]\left[D^2 - a^2\right]\left[D^2 - a^2 - \frac{s}{\nu}\right]\bar{w} = -\frac{a^2\alpha g\beta}{\kappa\nu}\bar{w}, \tag{5}$$

we define the dimensionless quantities

$$\hat{z} = \frac{z}{d}, \quad \hat{D} = dD, \quad \hat{a} = da, \quad \hat{s} = \frac{sd^2}{\kappa} \text{and} \quad \hat{\bar{w}} = \frac{d}{\kappa}\bar{w}. \tag{6}$$

Substituting these into (53) gives

$$\left[\frac{\hat{D}^2}{d^2} - \frac{\hat{a}^2}{d^2} - \frac{\hat{s}}{d^2}\right]\left[\frac{\hat{D}^2}{d^2} - \frac{\hat{a}^2}{d^2}\right]\left[\frac{\hat{D}^2}{d^2} - \frac{\hat{a}^2}{d^2} - \frac{\hat{s}}{d^2}\frac{\kappa}{\nu}\right]\frac{\kappa}{d}\hat{\bar{w}} = -\frac{\hat{a}^2\alpha g\beta}{d^2\kappa\nu}\frac{\kappa}{d}\hat{\bar{w}}. \tag{7}$$

Dividing by κ/d, multiplying by d^6 and dropping the hats for clarity, we get

$$\left[D^2 - a^2 - s\right]\left[D^2 - a^2\right]\left[D^2 - a^2 - s/\mathrm{P}\right]\bar{w} = -a^2\mathrm{R}\bar{w}. \tag{8}$$

in which $\mathrm{P} = \frac{\nu}{\kappa}$ and $\mathrm{R} = \frac{\alpha d^4 g\beta}{\kappa\nu}$.

Q3. As usual we perturb the basic state $\Phi_{\mathrm{B}} = \eta$ by writing $\Phi = \eta + \delta\phi$ with $|\delta| \ll 1$; and we seek a normal mode solution with

$$\phi = \cos(k\xi)F(\eta)\exp(st). \tag{9}$$

After substitution into the Eckhaus equation, we get

$$F'' + F\left[k^2R - k^4R - sR - k^2\right] = 0. \tag{10}$$

The solutions to this that satisfy the boundary conditions $\phi(0) = \phi(1) = 0$ are

$$F(\eta) = A\sin(n\pi\eta), \tag{11}$$

in which

$$n^2\pi^2 = R(k^2 - k^4 - s) - k^2 \quad \text{with} \quad n = 1, 2, 3\cdots \tag{12}$$

The flow first becomes unstable for mode $n = 1$ at $R = R_{\mathrm{c}}(k_{\mathrm{c}})$, where

$$R_{\mathrm{c}}(k_{\mathrm{c}}) = \frac{\pi^2 + k_{\mathrm{c}}^2}{k_{\mathrm{c}}^2 - k_{\mathrm{c}}^4}. \tag{13}$$

The minimum occurs when $\frac{dR_{\mathrm{c}}}{dk_{\mathrm{c}}} = 0$. To simplify the algebra define $\mu = k_{\mathrm{c}}^2$ and consider $\frac{dR_{\mathrm{c}}}{d\mu} = 0$, which leads to

$$\mu^2 + 2\pi^2\mu - \pi^2 = 0 \quad \text{and so} \quad \mu = k_{\mathrm{CM}}^2 = \pi\left[\sqrt{1 + \pi^{-2}} - 1\right]. \tag{14}$$