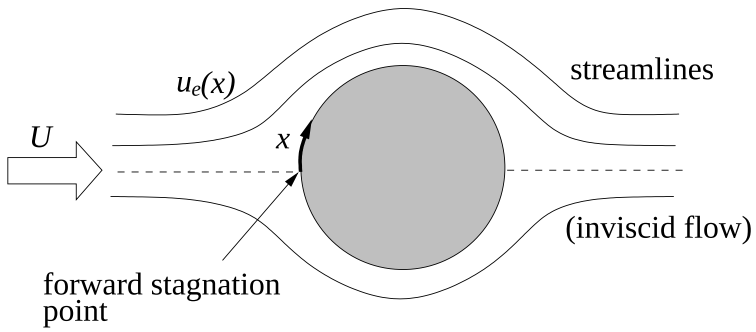
Laminar Boundary Layers

Answers to problem sheet 3: Exact Solutions and Separation.

1. Stagnation point

We seek a solution close to the forward stagnation point. (See sketch.)



The stream function Ψ is defined by,

$$u = \frac{\partial \Psi}{\partial y}, \qquad -v = \frac{\partial \Psi}{\partial x}.$$

We know that,

$$\frac{u}{U}, \frac{v}{U} Re^{1/2} = \text{functions}\left(\frac{x}{a}, \frac{y}{a} Re^{1/2}; \frac{u_e(x)}{U}\right), \quad \text{with } Re = \frac{Ua}{\nu}.$$

However, the boundary layer flow at any $x = x_1$ only has knowledge of its previous history $x < x_1$. At any curvilinear distance x from the nose, therefore, it cannot know the radius of the cylinder. As a consequence, we must replace x by a. We must also replace U by the the relevant velocity scale $u_e(x)$. Hence,

$$\frac{u}{u_e(x)}, \frac{v}{u_e(x)} Re_x^{1/2} = \text{functions}\left(\frac{y}{x} Re_x^{1/2}\right), \quad \text{with } \; Re_x = \frac{u_e x}{\nu}.$$

We have

$$\frac{u}{u_e} = \frac{1}{u_e} \frac{Re_x^{1/2}}{x} \frac{\partial \Psi}{\partial y'} = \frac{\partial}{\partial y'}\left(\frac{\Psi}{\sqrt{\nu u_e x}}\right),$$

so

$$\frac{\Psi}{\sqrt{\nu u_e x}} = \text{function}\left(\frac{y}{x} Re_x^{1/2}\right).$$

Similarly to the flat plate problem, we set:

$$\xi = x, \;\; \eta = \frac{y}{x} Re_x^{1/2} = \left(\frac{2U}{a\nu}\right)^{1/2} y, \;\; \text{and} \;\; \Psi = (\nu u_e x)^{1/2} f(\eta) = \left(\frac{2U\nu}{a}\right)^{1/2} \xi f(\eta).$$

The change of variables yields,

$$\left(\frac{\partial}{\partial x}\right)_y = \left(\frac{\partial}{\partial \xi}\right)_\eta \quad \text{and} \quad \left(\frac{\partial}{\partial y}\right)_x = \left(\frac{2U}{a\nu}\right)^{1/2} \left(\frac{\partial}{\partial \eta}\right)_\xi.$$

We can now calculate

$$u = \frac{\partial \Psi}{\partial y} = \left(\frac{2U}{\nu a}\right)^{1/2} \frac{\partial}{\partial \eta} \left\{ \left(\frac{2U\nu}{a}\right)^{1/2} \xi f(\eta) \right\},$$

i.e.

$$u = \frac{2U}{a} \xi f'.$$

Similarly,

$$-v = \frac{\partial \Psi}{\partial x} = \frac{\partial}{\partial \xi} \left(\left(\frac{2U\nu}{a}\right)^{1/2} \xi f(\eta) \right),$$

i.e.

$$v = -\left(\frac{2U\nu}{a}\right)^{1/2} f.$$

Expressed in the new variables, the convective operator is

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \frac{2U}{a}\left(\xi f' \frac{\partial}{\partial \xi} - f \frac{\partial}{\partial \eta}\right).$$

Hence, the BL momentum equation becomes,

$$\begin{aligned}
\left(\frac{2U}{a}\right)\left\{\xi f' \frac{\partial}{\partial \xi} - f\frac{\partial}{\partial \eta}\right\}\left(\frac{2U}{a}\xi f'\right) &= \left(\frac{2U}{a}\right)^2 \xi + \nu \left(\frac{2U}{a\nu}\right)^{1/2} \frac{\partial}{\partial \eta}\left\{\left(\frac{2U}{a\nu}\right)^{1/2} \frac{\partial}{\partial \eta}\left[\frac{2U}{a}\xi f'\right]\right\} \\
\textit{i.e.} \quad \left(\frac{2U}{a}\right)^2 \left\{\xi f'^2 - \xi f f''\right\} &= \left(\frac{2U}{a}\right)^2 \xi + \left(\frac{2U}{a}\right)^2 \xi f'''
\end{aligned}$$

i.e.

$$f''' + f f'' + 1 - f'^2 = 0,$$

with boundary conditions

$$\begin{aligned}
y = 0 \Rightarrow \eta = 0: \quad u = 0 \Rightarrow f'(0) &= 0 \\
v = 0 \Rightarrow f(0) &= 0 \\
\text{on the exterior edge of the boundary layer}: \ f'(\infty) &= 1.
\end{aligned}$$

2. Displacement and momentum thickness of the Blasius solution

From the notes on the Blasius boundary layer,

$$u = U f'(\eta) \quad \text{and} \quad \eta = \left(\frac{U}{2\nu x}\right)^{1/2} y,$$

so that

$$d\eta = \left(\frac{U}{2\nu x}\right)^{1/2} dy.$$

Hence,

$$\begin{aligned}
\delta^* &= \int_0^\infty \left(1 - \frac{u}{U}\right) dy \\
&= \left(\frac{2\nu x}{U}\right)^{1/2} \int_0^\infty (1 - f') d\eta \\
&= \left(\frac{2\nu x}{U}\right)^{1/2} \left[\eta - f\right]_0^\infty.
\end{aligned}$$

We know that $f(0) = 0$ and thus,

$$\delta^* = \left(\frac{2\nu x}{U}\right)^{1/2} \lim_{\eta \to \infty} (\eta - f).$$

Similarly,

$$\begin{aligned}
\theta &= \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \\
&= \left(\frac{2\nu x}{U}\right)^{1/2} \int_0^\infty f'(1 - f')d\eta \\
&= \left(\frac{2\nu x}{U}\right)^{1/2} \int_0^{f(\infty)} (1 - f')df, \\
\text{since } f'd\eta &= \frac{df}{d\eta}d\eta = df.
\end{aligned}$$

Integration by parts:

$g = 1 - f'$; $dg = -f''df$ and $h = f$; $dh = df$. We get

$$\int_0^{f(\infty)} (1 - f')df = [f(1 - f')]_0^{f(\infty)} - \int_0^{f(\infty)} (-f'')fdf,$$

and we have $f''' = -ff''$ (Blasius boundary layer solution), so

$$\int_0^{f(\infty)} (1 - f')df = -\int_0^{f(\infty)} f'''df,$$

since $f'(\infty) = 1$ and $f(0) = 0$. Hence,

$$\int_0^{f(\infty)} (1 - f')df = -\left[f''\right]_0^{f(\infty)} = f''(0),$$

because the boundary layer tends exponentially to the uniform outer stream, so that $f''(\infty) = 0$ (see III 14).

Finally, the momentum thickness becomes,

$$\theta = \left(\frac{2\nu x}{U}\right)^{1/2} f''(0).$$

3. Separation point:

Use the dimensional boundary layer equations expressed in terms of the pressure gradient dp/dx, *i.e.*

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}.
\end{aligned}$$

Differentiate the momentum equation equation with respect to y, *i.e.*

$$\frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} = \nu\frac{\partial^3 u}{\partial y^3}.$$

The continuity equation gives

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x},$$

so that the momentum equation becomes

$$u\frac{\partial^2 u}{\partial y \partial x} + v\frac{\partial^2 u}{\partial y^2} = \nu\frac{\partial^3 u}{\partial y^3}.$$

Differentiate the momentum equation again with respect to y, *i.e.*

$$\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y \partial x} + u\frac{\partial^3 u}{\partial y^2 \partial x} + \frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2} + v\frac{\partial^3 u}{\partial y^3} = \nu\frac{\partial^4 u}{\partial y^4}.$$

Use the continuity equation again to obtain,

$$\begin{aligned} \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y \partial x} + u\frac{\partial^3 u}{\partial y^2 \partial x} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} + v\frac{\partial^3 u}{\partial y^3} &= \nu\frac{\partial^4 u}{\partial y^4} \\ i.e. \qquad \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial^3 u}{\partial y^2 \partial x} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} + v\frac{\partial^3 u}{\partial y^3} &= \nu\frac{\partial^4 u}{\partial y^4}. \end{aligned}$$

On the body surface, *i.e.* at $y = 0$, $u = v = 0$ and $\partial u/\partial x = 0$, so the momentum equation reduces to,

$$\frac{1}{2}\frac{d}{dx}\left[\left(\frac{\partial u}{\partial y}\right)^2\right] = \nu\frac{\partial^4 u}{\partial y^4}.$$

If we assume that the right hand-side term is finite and non-zero at $y = 0$ then, by integrating the above equation in the vicinity of x_s, we find,

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} \propto (x_s - x)^{1/2},$$

i.e. there is a square-root singularity.