Laminar Boundary Layers

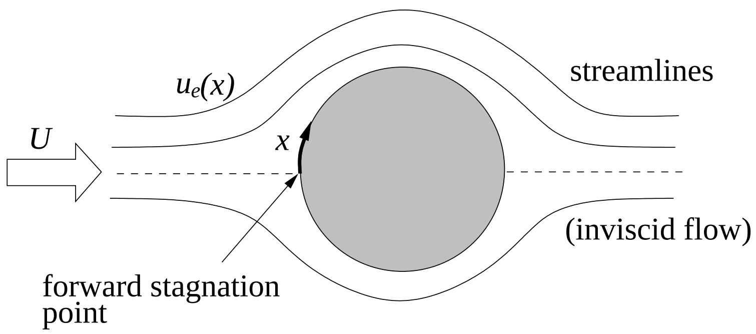
Problem sheet 3. Exact boundary layer solutions; separation.

1. Stagnation point

A cylinder of nose radius a is placed in a uniform stream of velocity U. Close to the forward stagnation point, the exterior velocity distribution at the outer edge of the boundary layer is given by

$$u_e = \frac{2Ux}{a},$$

in which x is the curvilinear distance measured round the cylinder's surface.



Extend the ideas on self-similarity of the boundary layer to suggest that

$$\frac{\Psi}{\sqrt{\nu u_e x}} = \text{function}\left(\frac{y}{x} Re_x^{1/2}\right), \quad Re_x = \frac{u_e x}{\nu}.$$

Hence, show that if $\xi = x$ then

$$u = \frac{2U}{a}\xi f', \quad u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} = \frac{2U}{a}\left(\xi f'\frac{\partial}{\partial \xi} - f\frac{\partial}{\partial \eta}\right),$$

and the boundary layer momentum equation becomes

$$f''' + ff'' + 1 - f'^2 = 0; \quad f(0) = f'(0) = 0, \; f'(\infty) = 1.$$

Note that in this flow geometry the scaling variable $yRe_x^{1/2}/x$ actually turns out not to depend on x (since $Re_x \propto x^2$ here) so that the boundary layer thickness near a forward stagnation point is constant. (In contrast, for a sharp-nosed body, *e.g.* the flat plate, the boundary layer thickness is zero at $x = 0$.)

As we saw in section 4.2 of the notes, other self-similar solutions exist in which $u_e \propto x^m$, with a momentum boundary layer equation of the form:

$$f''' + ff'' + \frac{2m}{1+m}(1 - f'^2) = 0; \; f(0) = f'(0) = 0, \; f'(\infty) = 1.$$

The cases $m = 0$ (Blasius) and $m = 1$ (the stagnation flow considered here) are then seen as special cases of the general case $u_e \propto x^m$. The inviscid solution with $u_e \propto x^m$ turns out to be that for flow past a straight-sided wedge of angle $2\pi m/(m+1)$. Thus, the value of m specifies the wedge angle and therefore the body shape.

2. Displacement and momentum thicknesses

Deduce equations 104 and 105 in the notes by showing that:

$$\begin{aligned}\delta^* &= \int_0^\infty \left(1 - \frac{u}{U}\right) dy \\ &= \left(\frac{2\nu x}{U}\right)^{1/2} \lim_{\eta\to\infty} (\eta - f),\end{aligned}$$

and

$$\begin{aligned}\theta &= \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \\ &= \left(\frac{2\nu x}{U}\right)^{1/2} \int_0^{f(\infty)} (1 - f')df \\ &= \left(\frac{2\nu x}{U}\right)^{1/2} f''(0).\end{aligned}$$

For the second part use the Blasius equation (93 in the notes) in the form $ff'' = -f'''$.

3. Separation point

Use the dimensional boundary layer equations to consider flow in the immediate vicinity of a separation point $x = x_s$, where $(\partial u/\partial y)_{y=0} = 0$. Show that

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} \propto (x_s - x)^{1/2},$$

when x is close to x_s, provided that $(\partial^4 u/\partial y^4)_{y=0}$ is finite and non-zero at $x = x_s$.