Reference sheet

In Sec. 3 we derived the boundary layer equations for 2D incompressible flow with constant viscosity. In Sec. 4 we will use them to study various boundary layer phenomena. We collect them here, in their various forms, for convenient reference.

Boundary layer equations

In **dimensionless form**, with the transformation variables

$$x' = \frac{x}{L},\ y' = Re^{1/2}\frac{y}{L},\ u' = \frac{u}{U},\ v' = Re^{1/2}\frac{v}{U} \ \text{ and } \ \Psi' = \frac{\Psi}{\sqrt{\nu UL}} \ \text{ with } \ Re = \frac{UL}{\nu} \tag{65}$$

Continuity

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0 \tag{66}$$

Momentum

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = u'_e\frac{du'_e}{dx'} + \frac{\partial^2 u'}{\partial y'^2} \tag{67}$$

BCs

$$u' = v' = 0 \ \text{ at } \ y' = 0; \ \ u' \to u'_e(x') \ \text{ as } \ y' \to \infty \tag{68}$$

In **dimensional form**

Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{69}$$

Momentum

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{70}$$

BCs

$$u = v = 0 \ \text{ at } \ y = 0; \ \ u \to u_e(x) \ \text{ as } \ y \to \infty \tag{71}$$

Notice that the dimensionless form is actually very similar: just replace $x \to x'$, $u \to u'$ *etc.*, and remove the prefactor ν from the viscous term.

The stream function

We can also express the BL equations in terms of the stream function Ψ, defined by

$$u = \frac{\partial \Psi}{\partial y},\ v = -\frac{\partial \Psi}{\partial x}, \tag{72}$$

which automatically satisfies the continuity equation. We are then left with just

Momentum

$$\frac{\partial \Psi}{\partial y}\frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial y^2} = u_e\frac{du_e}{dx} + \nu\frac{\partial^3 \Psi}{\partial y^3} \tag{73}$$

BCs

$$\frac{\partial \Psi}{\partial y} = \frac{\partial \Psi}{\partial x} = 0 \text{ at } y = 0; u = \frac{\partial \Psi}{\partial y} \to u_e(x) \text{ as } y \to \infty \tag{74}$$

The dimensionless form again simply has $x \to x'$, $u \to u'$ *etc.*, with the prefactor ν removed from the viscous term $\nu\frac{\partial^3 \Psi}{\partial y^3}$.

Reference sheet (continued)

Forces on the body

At the end of Sec. 3.4, we defined the skin friction coefficient

$$c_f = \frac{(\Pi_{xy})_{y=0}}{\frac{1}{2}\rho U^2} = \frac{\mu(\partial u/\partial y)_{y=0}}{\frac{1}{2}\rho U^2}. \tag{75}$$

In dimensionless form

$$c_f Re^{1/2} = 2\left(\frac{\partial u'}{\partial y'}\right)_{y'=0} \tag{76}$$

Boundary layer thickness

Two measures of this will be discussed in Sec. 4.1 below:

- **Displacement thickness**

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{u_e}\right) dy, \tag{77}$$

- **Momentum thickness**

$$\theta = \int_0^\infty \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) dy. \tag{78}$$