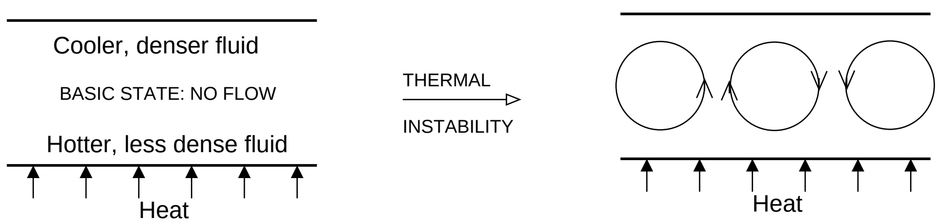
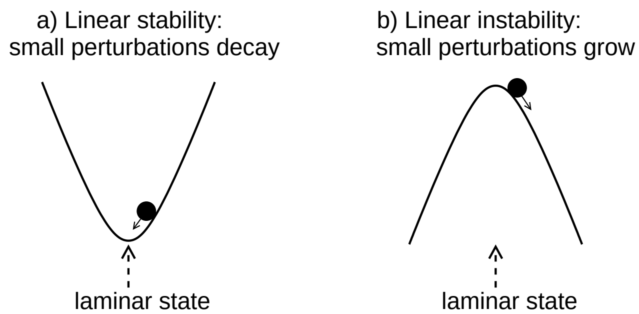
1 Introduction

The basic equations of fluid dynamics admit some simple laminar flow states as stationary solutions. Examples include simple Couette flow between rotating cylinders; and a horizontal layer of stationary fluid heated from below. When the parameters governing these flows (the speed of cylinder rotation; the applied thermal gradient, *etc.*) exceed certain critical values, the laminar states can go unstable. More complicated flow states then ensue: roll-states in these sketches.



Part I of the course concerns the initial onset of such instabilities, studied via the dynamics of tiny perturbations to the basic laminar flow state. If these perturbations grow in time the laminar flow is said to be linearly unstable, as sketched conceptually below. If they decay, the laminar state is linearly stable. The analogy of the sketch is less superficial than might appear, as our study of normal modes below will show.



For larger perturbations, nonlinear effects become important. These will be studied in part II of the course.

1.1 The basic procedure of linear stability analysis

We now outline the basic procedure involved in a linear stability analysis. This will be used repeatedly throughout the course: for thermal convection, centrifugal instabilities, and shear flow instabilities in Secs. 2, 3 and 4 respectively. In this section we use quite a compact notation, which might seem opaque on a first reading. However as the course progresses it can be referred back to as the basic template followed in each case. In the next section 1.2 we give a concrete example in the context of a very simple model. It might help (on a second reading) to compare each subsection 1.1.1 to 1.1.5 with its counterpart in 1.2.1 to 1.2.5.

1.1.1 Specify the governing equations and boundary conditions

First, the full system of nonlinear equations is specified. For most of the phenomena studied in this course these are the Navier-Stokes equations of incompressible fluid flow, subject in Sec. 2 to a slight extension to allow for thermal effects. These have been covered in previous courses, and are summarised here as a reminder only. A fuller revision can be found, *e.g.*, "Physical Fluid Dynamics" by Tritton.

Denoting the flow field by $\mathbf{u}$ and the pressure by p, for a viscous incompressible fluid we have

Mass continuity

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

and the **momentum equations**

$$\rho\left[\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = -\nabla p + \eta\nabla^2\mathbf{u} + \rho\,\mathbf{g}. \tag{2}$$

Here ρ is the fluid's density, assumed constant; η its viscosity; and $\mathbf{g}$ the acceleration due to gravity. The 3 terms on the RHS of Eqn. 2 arise from pressure gradients, viscous forces and gravitational body forces respectively. The LHS also has the dimensions of force, and is correspondingly referred to as the inertial force.

If the equations are initially given in coordinate-free form, as here, components in the relevant coordinate system must then be extracted. The boundary conditions for the geometry are also specified.

1.1.2 Find the base state

A basic laminar flow field $\mathbf{u}_\mathrm{B}(\mathbf{x})$ and pressure field $p_\mathrm{B}(\mathbf{x})$ that form a time-independent solution

$$\mathbf{N}\{\mathbf{u}_\mathrm{B}, p_\mathrm{B}, \lambda\} = \mathbf{0} \tag{3}$$

of the nonlinear equations and boundary conditions is found. $\mathbf{x}$ denotes space. (For thermal convection, Sec. 2, the temperature field $T_\mathrm{B}(\mathbf{x})$ must obviously also be specified, though we disregard it for now.) In any experiment, the basic state – and its stability properties – depends on one or more external control parameters: the applied thermal gradient in convection problems; the applied rotation rate at the boundary of cylindrical Couette flow, *etc.* For the moment we represent the control parameter by a generalised variable λ, whose physical identity we do not specify.

1.1.3 Add a small perturbation

We now subject the base state to a small perturbation. Experimentally, perturbations arise naturally via thermal or mechanical background noise:

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}_{\mathrm{B}}(\mathbf{x}) + \delta\tilde{\mathbf{u}}(\mathbf{x},t), \quad \text{and} \quad p(\mathbf{x},t) = p_{\mathrm{B}}(\mathbf{x}) + \delta\tilde{p}(\mathbf{x},t). \tag{4}$$

Here δ is a small constant prefactor to $\tilde{\mathbf{u}}(\mathbf{x},t), \tilde{p}(\mathbf{x},t)$, with $|\delta| \ll 1$.

In what follows, we will be interested in the following questions. (Q1) For any value of the control parameter λ, is the basic laminar state linearly stable or unstable, *i.e.* do the perturbations decay or grow in time? (Q2) What is the threshold value of λ at which the laminar state first becomes unstable? (Q3) At the onset of instability, what is the spatial form of the unstable perturbations, and how fast do they grow?

1.1.4 Linearise the equations

To answer these questions, we substitute the perturbed forms (4) into the governing equations (1, 2) and expand these equations about the base state in increasing powers of the perturbation's amplitude δ. Neglecting terms $O(\delta^2)$ and higher, we then have a linearised equation set governing the dynamics of the perturbations $\tilde{\mathbf{u}}(\mathbf{x},t), \tilde{p}(\mathbf{x},t)$.

1.1.5 Solve the linearised equations using normal modes

These linearised equations are usually studied via a normal mode analysis:

$$\tilde{\mathbf{u}}(\mathbf{x},t) = \int ds\, \hat{\mathbf{u}}_s(\mathbf{x}) \exp(st) \tag{5}$$

with a similar form for the pressure, which we do not bother to write here. By such means, they reduce to the form of an eigenvalue problem for each individual mode:

$$\mathcal{L}\{\hat{\mathbf{u}}_s(\mathbf{x}); s, \lambda\} = \mathbf{0}. \tag{6}$$

Here $\mathcal{L}$ is a linear operator, which typically has the following properties. (i) It contains spatial gradients ∇, ∇^2, *etc.* (ii) It depends on the base state $\mathbf{u}_{\mathrm{B}}(\mathbf{x})$. (iii) In curved flow geometries, it can itself depend explicitly on space $\mathbf{x}$.

The eigenvalues s and corresponding eigenfunctions $\hat{\mathbf{u}}_s(\mathbf{x})$ of $\mathcal{L}$ are then calculated. In general, the eigenvalues are complex:

$$s = s_{\mathrm{r}} + is_{\mathrm{i}} \quad \text{so that} \quad \exp(st) = \exp(s_{\mathrm{r}}t)[\cos(s_{\mathrm{i}}t) + i\sin(s_{\mathrm{i}}t)]. \tag{7}$$

- An eigenmode with $s_{\mathrm{r}} < 0$ is linearly stable. Any corresponding contribution $\hat{\mathbf{u}}_s(\mathbf{x})$ to the small perturbation decays exponentially in time.
- An eigenmode with $s_{\mathrm{r}} = 0$ is neutrally stable.
- An eigenmode with $s_{\mathrm{r}} > 0$ is linearly unstable. Any corresponding contribution $\hat{\mathbf{u}}_s(\mathbf{x})$ to the small perturbation grows (initially) exponentially in time.

If any eigenvalue has $s_{\mathrm{r}} > 0$, the laminar state will be unstable. So for the laminar state to be stable, all the eigenvalues of $\mathcal{L}$ must obey $s_{\mathrm{r}} < 0$. This answers Q1 above. The threshold value of λ at which the base state first goes unstable is where the largest real part changes from negative to positive (Q2). In the unstable regime, this most unstable eigenvalue determines how fast the perturbation grows; its corresponding eigenfunction determines their spatial form (Q3).

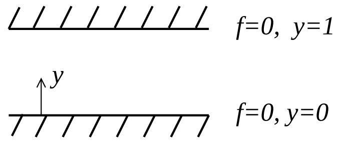
1.2 A worked example

1.2.1 Governing equations and boundary conditions

Consider the equation

$$\partial_t f(y,t) = f - f^2 + \frac{1}{\lambda}\,\partial_y^2 f \tag{8}$$

for the following geometry and boundary conditions:



Here λ will be used as the control parameter to be varied externally.

1.2.2 Base state

Trivial stationary ($\partial_t f = 0$) basic states are homogeneous here, independent of y.

$$f_{\rm B}(y) = 0 \ \text{ and } \ f_{\rm B}(y) = 1. \tag{9}$$

Of course only one of these, $f_{\rm B} = 0$, satisfies the boundary conditions.

1.2.3 Small perturbation

We now add a small perturbation to the basic state, writing

$$f(y,t) = f_{\rm B}(y) + \delta \tilde{f}(y,t). \tag{10}$$

1.2.4 Linearise the equations

Now substitute the perturbed form (10) into the governing equation (8):

$$\begin{aligned}\partial_t(f_{\rm B} + \delta\tilde{f}) &= f_{\rm B} + \delta\tilde{f} - (f_{\rm B} + \delta\tilde{f})^2 + \frac{1}{\lambda}\partial_y^2(f_{\rm B} + \delta\tilde{f}) \\ &= f_{\rm B} - f_{\rm B}^2 + \frac{1}{\lambda}\partial_y^2 f_{\rm B} + \delta\left[\tilde{f} - 2f_{\rm B}\tilde{f} + \frac{1}{\lambda}\partial_y^2\tilde{f}\right] + O(\delta^2)... \end{aligned} \tag{11}$$

On the second line we have grouped terms in successive powers of δ, which is a small parameter for the tiny perturbations of interest.

• At $O(\delta^0)$ we have $\partial_t f_{\rm B} = f_{\rm B} - f_{\rm B}^2 + \frac{1}{\lambda}\partial_y^2 f_{\rm B}$. This is just the trivial equation obeyed by the basic state, as already solved above.

• **At O(δ), which is the interesting part, we have**

$$\partial_t\tilde{f} = \tilde{f} - 2f_{\rm B}\tilde{f} + \frac{1}{\lambda}\partial_y^2\tilde{f} \tag{12}$$

For a given base state $f_{\rm B}(y)$, this linearised equation governs the dynamics of the small perturbations $\tilde{f}(y,t)$. Below we will choose to consider the basic state $f_{\rm B} = 0$, for which

$$\partial_t\tilde{f} = \tilde{f} + \frac{1}{\lambda}\partial_y^2\tilde{f} \tag{13}$$

• Terms $O(\delta^2)$ and higher are neglected.

1.2.5 Solve the linearised equations using normal modes

We now express $\tilde{f}(y,t)$ as a sum of normal modes, each of which has the general form

$$\tilde{f}(y,t) = F(y)\exp(st). \tag{14}$$

Substituting this into the linearised equation (13) we get

$$sF = F + \frac{1}{\lambda}F''. \tag{15}$$

Given the boundary conditions $f = 0$ at $y = 0, 1$, the function $F(y)$ must clearly obey boundary conditions $F(0) = F(1) = 0$. Therefore we have solutions of the form

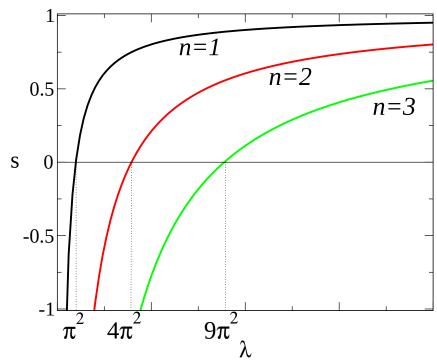
$$F(y) = c.\sin(n\pi y). \tag{16}$$

Substituting this into Eqn. 15, we get

$$s = 1 - \frac{n^2\pi^2}{\lambda}. \tag{17}$$

So in this case the eigenvalues are real. As seen in the below sketch, the *nth* mode is

- decaying $s < 0$ for $\lambda < n^2\pi^2$,
- neutral $s = 0$ for $\lambda = n^2\pi^2$,
- growing $s > 0$ for $\lambda > n^2\pi^2$.



So the basic state is linearly stable (all eigenvalues negative) for $\lambda < \pi^2$. As λ is increased, the basic state first becomes unstable at $\lambda = \pi^2$. In the unstable regime $\lambda > \pi^2$, the most unstable mode is the one with the most positive eigenvalue: *i.e.* mode $n = 1$, which has $s = 1 - \pi^2/\lambda$. The physical form of this fastest growing mode is then given by

$$\tilde{f}(y,t) = c.\sin(\pi^2 y)\exp\left[\left(1 - \frac{\pi^2}{\lambda}\right)t\right]. \tag{18}$$