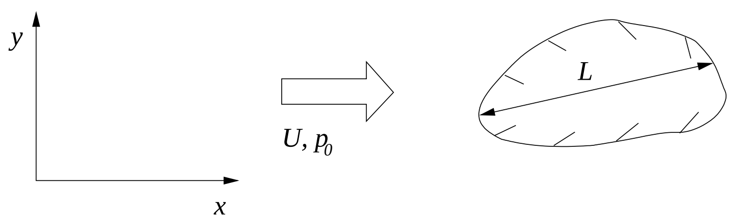
2 Introducing the boundary layer

In this section, we introduce the concept of the boundary layer. We start in Sec. 2.1 by discussing the relative importance of viscous and inertial forces in the fluid. This is encoded in a non-dimensional parameter called the Reynolds number, Re. The regime of interest here (relevant to everyday flows of most simple fluids) is that of large Re, where inertia dominates. But the limit $Re \to \infty$ must be taken carefully, Sec. 2.2: although inertia indeed dominates over most of the flow, at any solid surface there exists a thin boundary layer in which viscous forces remain important. Over this layer, the fluid velocity decreases precipitously from its bulk flow value to zero ("no-slip") at the solid surface, Sec. 2.3.

2.1 Viscous vs inertial forces

The relative importance of viscous and inertial forces is determined by the typical velocity and length scales of the flow, as we will now show. Consider steady 2D incompressible flow with constant viscosity over a solid body of characteristic length scale L, with a uniform free stream far from the body of velocity U in the x direction.



As a first step, we cast the flow equations into dimensionless form by expressing lengths, velocities and pressure in units of the natural scales L, U and ρU^2:

$$x'_i = \frac{x_i}{L}, \quad u'_j = \frac{u_j}{U}, \quad P' = \frac{P}{\rho U^2}. \tag{25}$$

By this transformation, Eqns. 22 to 24 become

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0 \tag{26}$$

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\frac{\partial P'}{\partial x'} + \frac{1}{Re}\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}\right) \tag{27}$$

$$u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} = -\frac{\partial P'}{\partial y'} + \frac{1}{Re}\left(\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right), \tag{28}$$

in which appears the non-dimensional

$$\textbf{Reynolds number} \qquad Re = \frac{UL\rho}{\mu} = \frac{UL}{\nu}. \tag{29}$$

We have defined the dynamic viscosity $\nu = \mu/\rho$ for convenience. These equations must be solved subject to the following boundary conditions (BCs):

- On the body's surface, $u' = 0$, $v' = 0$ (no permeation and no slip, Sec. 1.5).
- Far from the body the free stream is undisturbed: $u' \to 1$, $v' \to 0$.

2.1.1 Dynamical similarity

Eqns. 26 to 28 lead to the important concept of *dynamical similarity*, as follows. For a given, fixed Re and a given, fixed set of BCs, we have a given, fixed solution

$$(u', v') = \mathrm{Functions(x', y')}, \text{ for fixed BCs, Re.} \tag{30}$$

Imagine now doubling L (*i.e.*, considering a larger object), for fixed BCs and Re. What does this mean? To preserve the BCs ($u' = 0, v' = 0$ on the line bounding the object), the shape and orientation clearly must not change. The new object must be *geometrically similar* to the first, *i.e.*, just a scaled up version of it. To keep Re fixed, we halve U. The key point is that the scaled solution $(u', v') = \mathrm{Functions(x', y')}$ is then unchanged, despite the changes in L and U. The ratio of each velocity component to U is the same at points with the same scaled position (x', y'). We thus only need to compute $\mathrm{Functions(x', y')}$ once, and it describes (after rescaling) the flow round all geometrically similar objects at that given value of Re. This is called dynamical similarity.

2.1.2 The Reynolds number

We have just seen that the (scaled) flow field is determined by Re. What does Re mean physically? Looking back at Eqns. 27 and 28, we see that it describes the relative sizes of inertial ($\rho \mathbf{v} \cdot \nabla \mathbf{v}$) and viscous ($\mu \nabla^2 \mathbf{v}$) forces:

$$Re \sim \frac{\text{inertial forces}}{\text{viscous forces}}. \tag{31}$$

At low $Re \to 0$, inertial forces are negligible. The N–S equation then reduces to that of "creeping" flow, in which pressure gradients are directly balanced by viscous forces[2]

$$0 = -\nabla' P' + \frac{1}{Re} \nabla'^2 \mathbf{v}'. \tag{32}$$

This regime is not relevant to this course.

2.2 Apparent singularity of the limit $Re \to \infty$

At high Re, inertial forces dominate. One might expect in this case simply to be able to ignore viscous forces, as we (correctly) ignored inertia in Eqn. 32. Doing so, the N–S equations reduce to the **Euler equations** of **inviscid flow**, in which the fluid in each element has an acceleration set by the pressure gradient:

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0 \tag{33}$$

$$u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} = -\frac{\partial P'}{\partial x'} \tag{34}$$

$$u' \frac{\partial v'}{\partial x'} + v' \frac{\partial v'}{\partial y'} = -\frac{\partial P'}{\partial y'}. \tag{35}$$

[2]Note that the pressure term must *always* be retained so that the number of unknowns matches the number of equations. The force balance and continuity equations represent one vector equation and one scalar equation in one vector quantity, $\mathbf{v}$, and one scalar, p.

However this turns out to be too naive an approach, for the following reason. We stated (just before Sec. 2.1.1) that the flow field must obey **two BCs** at the object's surface: no permeation and no slip. In contrast, the inviscid flow equations contain only **first order** derivatives. **We therefore cannot find a solution that satisfies both BCs**. We say that the limit $Re \to \infty$ is **singular**, because $1/Re$ multiplies the highest order derivative in the N–S equations.

2.3 Resolution of this singularity: the boundary layer

How do we resolve this? An obvious (but wrong) suggestion is simply to throw away the no-slip condition. This would leave just one BC (no permeation), matching the order of the inviscid equations. But experiment shows that the no-slip condition continues to apply at the solid surface, even at high Re. We are therefore forced to conclude (Prandtl, 1904) that viscous forces remain significant in a thin **boundary layer** (BL) surrounding the surface, so that the order of the equations still matches the number of BCs, even as $Re \to \infty$. The reasoning that led us to neglect viscous forces breaks down in this region because the characteristic length scale is not now L, but the small layer thickness, $\delta \sim Re^{-1/2} L \ll L$ (to be derived in Sec. 3).

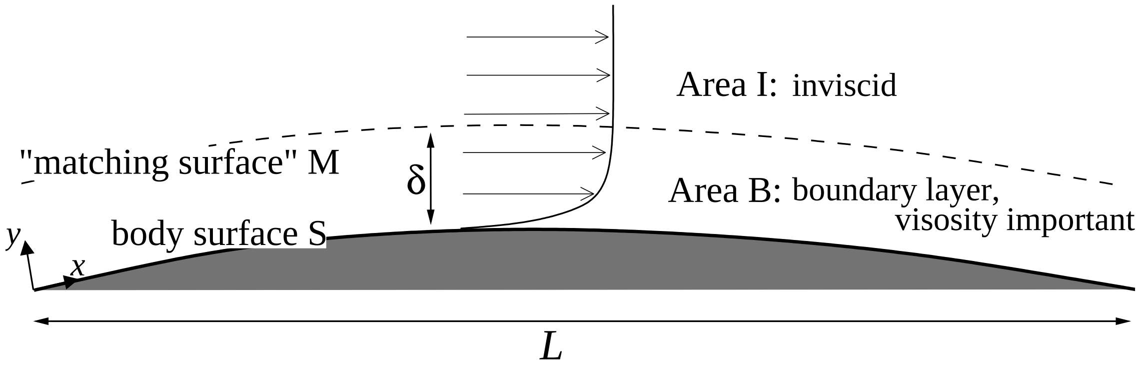


Figure 4: Sketch of the boundary layer. (Note that in general the thickness δ will vary along the surface. This is not shown here, for simplicity.)

Outside the BL (area I in Fig. 4) the flow is truly inviscid, and can be calculated using Euler's Eqns. 33 to 35. Inside the BL (area B), we use the boundary layer equations to be derived in Sec. 3. On the body side of the BL, we apply the no-slip condition on the body surface (S in Fig. 4). For the BC on the inviscid side of the BL (surface M, defined precisely in Sec. 3), we take the slipping velocity given by the inviscid calculation: we *match* the boundary layer solution onto the outer inviscid one. In this way, the first order inviscid (outer) equations are absolved of the no-slip BC. This is instead allowed for by the steep velocity gradient across the boundary layer, which generates significant viscous forces.