

## 2 Thermal instabilities: Bénard convection

Here we consider a horizontal layer of fluid heated from below. The base state is one of no flow, with a negative thermal gradient upwards through the layer. The cooler fluid near the top of the layer is denser than the warmer fluid underneath it. For a large enough thermal gradient this buoyancy effect causes an overturning instability, leading to convection rolls. For smaller thermal gradients, it is counteracted by the stabilising effects of viscosity and thermal conductivity, and the base state remains stable.

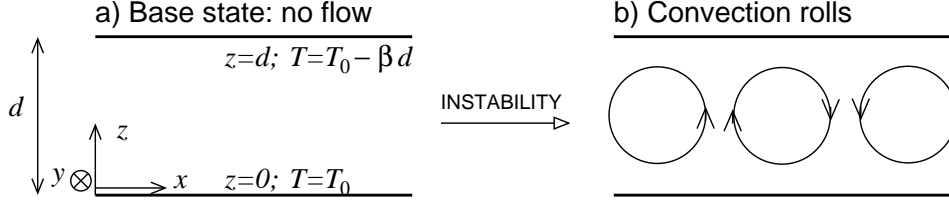


Figure 1: a) base state – no flow and a vertical temperature gradient; b) convection rolls.

### 2.1 Governing equations: the Boussinesq approximation

Consider a fluid in which the temperature has small variations  $\Delta T$  about a constant value  $T_0$ , leading to variations in the local fluid density:

$$T = T_0 + \Delta T, \quad \text{and} \quad \rho = \rho_0 + \Delta\rho. \quad (19)$$

To allow for these, we need to extend the basic equations (1 and 2) slightly. In principle this is quite complicated. However in most cases the variations in temperature  $\Delta T$ , and so in density  $\Delta\rho$ , are small ( $|\Delta\rho/\rho_0| \ll 1$ ). We then use the so called “Boussinesq approximation”, which takes

$$\Delta\rho = -\alpha \cdot \Delta T \cdot \rho_0 \quad (20)$$

in the body force  $\rho\mathbf{g}$  of Eqn. 2. Here  $\alpha$  is the (constant) coefficient of thermal expansion. In all other terms the density is assumed constant,  $\rho = \rho_0$ . Thus we have:

**Mass continuity** (unaltered)

$$\nabla \cdot \mathbf{u} = 0. \quad (21)$$

**Momentum equations**, in which only the body force changes

$$\rho_0 [\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}] = -\nabla p + \eta \nabla^2 \mathbf{u} + \rho_0 (1 - \alpha \Delta T) \mathbf{g}. \quad (22)$$

Finally, we use the **thermal diffusion equation** to describe temperature variations:

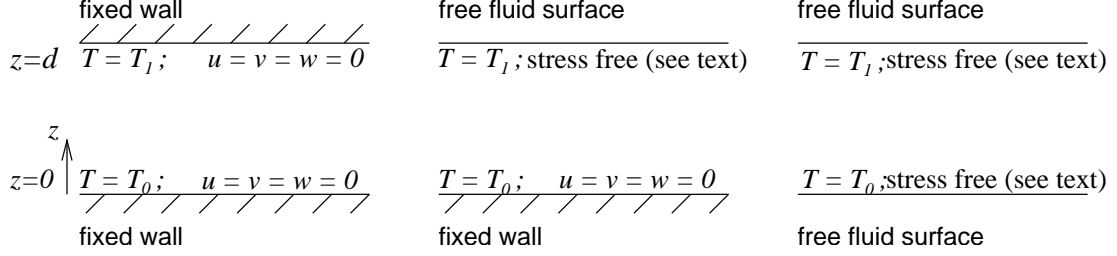
$$\partial_t T + (\mathbf{u} \cdot \nabla) T = \kappa \nabla^2 T \quad (23)$$

in which  $\kappa$  is the (constant) coefficient of thermal diffusion. Eqns. 21, 22 and 23 are the Boussinesq equations, and form the full nonlinear equation set for the remainder of this section. The gravitational field is taken to act vertically downwards:

$$\mathbf{g} = (0, 0, -g)^T \quad \text{where} \quad g = 9.81 \text{ms}^{-2}. \quad (24)$$

### 2.1.1 Geometry and boundary conditions

The fluid can either be confined between two fixed walls, have its upper surface free to the air, or have both surfaces free. The last case is unrealistic, but will be the one considered below because it is the most easily handled analytically.



In either case (fixed wall or free surface) the temperature is held constant along each boundary to give an applied thermal gradient  $-\beta$ :

$$T = T_0 \quad (\text{lower boundary}); \quad T = T_1 = T_0 - \beta d \quad (\text{upper boundary}). \quad (25)$$

The boundary conditions on the velocity are as follows. At a fixed wall we have

$$u = v = 0 \quad (\text{no slip}); \quad \text{and} \quad w = 0 \quad (\text{no permeation}). \quad (26)$$

At a free surface it can be shown that

$$w = \partial^2 w / \partial z^2 = 0 \quad (\text{stress free condition}). \quad (27)$$

## 2.2 The base state

Here we look for a stationary base state in which there exists a negative temperature gradient upwards through the fluid, but no fluid flow. This is given by:

$$\mathbf{u}_B = \mathbf{0}, \quad (28)$$

$$T_B = T_0 - \beta z. \quad (29)$$

These can easily be shown to obey Eqns. 21 and 23, together with the boundary conditions. The heating at the bottom surface has thus resulted in a linear variation of the temperature upwards through the fluid. In turn this gives a linear density variation  $\rho = \rho_0(1 + \alpha\beta z)$ , with cooler denser fluid above warmer less dense fluid. The only non-trivial component of the momentum equation, Eqn. 22, is then:

$$\frac{dp_B}{dz} = -g\rho_0(1 + \alpha\beta z). \quad (30)$$

In principle, we can easily solve this for the vertical pressure distribution

$$p_B = p_B(z) \quad (31)$$

though we will not need this below.

## 2.3 Add a small perturbation

We now subject this base state to an infinitesimal perturbation:

$$\mathbf{u} = \mathbf{u}_B + \delta \tilde{\mathbf{u}}(\mathbf{x}, t) \quad , \quad T = T_B(z) + \delta \tilde{T}(\mathbf{x}, t) \quad , \quad p = p_B(z) + \delta \tilde{p}(\mathbf{x}, t), \quad (32)$$

in which  $\delta$  is a small constant prefactor. Recall that the basic velocity  $\mathbf{u}_B = \mathbf{0}$ .

## 2.4 Linearise the governing equations

We now substitute (32) into the basic equations 21, 22 and 23 following the linearisation procedure described in detail in Sec. 1.2.4. Collecting terms  $O(\delta^0)$ , we get the equations governing the base flow, which has already been found. Collecting the linear terms  $O(\delta^1)$  we get the linearised equations:

**Continuity**

$$\nabla \cdot \tilde{\mathbf{u}} = 0, \quad (33)$$

**Momentum**

$$\rho_0 \partial_t \tilde{\mathbf{u}} = -\nabla \tilde{p} + \eta \nabla^2 \tilde{\mathbf{u}} - \rho_0 \alpha \tilde{T} \mathbf{g}, \quad (34)$$

**Thermal diffusion**

$$\partial_t \tilde{T} + (\tilde{\mathbf{u}} \cdot \nabla) T_B = \kappa \nabla^2 \tilde{T}. \quad (35)$$

All terms  $O(\delta^2)$  or higher are neglected. Linearisation procedures such as these will crop up repeatedly throughout the course. Do this one as an exercise!

### 2.4.1 Linearised equations in component form

In component form, with  $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v}, \tilde{w})^T$  and  $\mathbf{g} = (0, 0, -g)^T$  we have

**Continuity**

$$\partial_x \tilde{u} + \partial_y \tilde{v} + \partial_z \tilde{w} = 0 \quad (36)$$

**Momentum**

$$\partial_t \tilde{u} = -\frac{1}{\rho_0} \partial_x \tilde{p} + \nu \nabla^2 \tilde{u} \quad (37)$$

$$\partial_t \tilde{v} = -\frac{1}{\rho_0} \partial_y \tilde{p} + \nu \nabla^2 \tilde{v} \quad (38)$$

$$\partial_t \tilde{w} = -\frac{1}{\rho_0} \partial_z \tilde{p} + \nu \nabla^2 \tilde{w} + \alpha g \tilde{T} \quad (39)$$

**Thermal diffusion**

$$\partial_t \tilde{T} - \beta \tilde{w} = \kappa \nabla^2 \tilde{T}. \quad (40)$$

For convenience, we have divided the momentum equations across by  $\rho_0$  and defined the coefficient of kinematic viscosity  $\nu = \eta/\rho$ .

### 2.4.2 Simplifying the linearised equations

Above we have five PDEs (Eqns. 36 to 40) for the five functions  $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}, \tilde{T}$ . Our strategy in this section is to simplify these to two PDEs in the two functions  $\tilde{w}, \tilde{T}$ . Taking  $\partial_x(37)+\partial_y(38)+\partial_z(39)$  and using  $\partial_x\tilde{u} + \partial_y\tilde{v} + \partial_z\tilde{w} = 0$  (Eqn. 36), we get

$$0 = -\frac{1}{\rho_0}\nabla^2\tilde{p} + \alpha g \partial_z\tilde{T}, \quad (41)$$

and will not need to consider  $\tilde{u}, \tilde{v}$  further. We note that this equation contains  $\nabla^2\tilde{p}$ . In order to eliminate  $\tilde{p}$ , we now take  $\nabla^2(39)$  to get

$$\partial_t\nabla^2\tilde{w} = -\frac{1}{\rho_0}\partial_z\nabla^2\tilde{p} + \nu\nabla^4\tilde{w} + \alpha g\nabla^2\tilde{T}, \quad (42)$$

and substitute  $\frac{1}{\rho_0}\nabla^2\tilde{p}$  from Eqn. 41, giving

$$\partial_t\nabla^2\tilde{w} = -\alpha g \partial_z^2\tilde{T} + \nu\nabla^4\tilde{w} + \alpha g\nabla^2\tilde{T}. \quad (43)$$

Tidying this up, we have

$$\partial_t\nabla^2\tilde{w} = \alpha g \left( \partial_x^2 + \partial_y^2 \right) \tilde{T} + \nu\nabla^4\tilde{w}. \quad (44)$$

Recalling our original Eqn. 40

$$\partial_t\tilde{T} - \beta\tilde{w} = \kappa\nabla^2\tilde{T} \quad (45)$$

we now have two PDEs, Eqns. 44 and 45, in the two unknown functions  $\tilde{w}$  and  $\tilde{T}$ . We now seek solutions of these two equations via a normal mode analysis.

## 2.5 Solve the linearised equations using normal modes

### 2.5.1 Separability

The form of Eqns. 44, 45 allows us to seek separable normal mode solutions:

$$\tilde{w}(x, y, z, t) = f(x, y)\bar{w}(z) \exp(st) \quad \text{and} \quad \tilde{T}(x, y, z, t) = f(x, y)\theta(z) \exp(st). \quad (46)$$

Substituting these into Eqn. 44 and 45 gives respectively

$$s \left[ \left\{ \left( \partial_x^2 + \partial_y^2 \right) f \right\} \bar{w} + f\bar{w}'' \right] = \alpha g \left( \partial_x^2 + \partial_y^2 \right) f \cdot \theta + \nu \left[ \left( \partial_x^2 + \partial_y^2 + \partial_z^2 \right)^2 f \cdot \bar{w} \right], \quad (47)$$

and

$$s \cdot f \cdot \theta - \beta \cdot f \cdot \bar{w} = \kappa \left\{ \left( \partial_x^2 + \partial_y^2 \right) f \cdot \theta + f\theta'' \right\}. \quad (48)$$

Dividing Eqn. 48 across by  $f$  we have

$$s\theta(z) - \beta\bar{w}(z) = \kappa\theta(z) \frac{1}{f(x, y)} \left( \partial_x^2 + \partial_y^2 \right) f(x, y) + \kappa\theta''(z). \quad (49)$$

Looking at the  $x, y$  and  $z$  dependencies of the various terms in Eqn. 49, we see that we can only seek separable solutions when, for some constant  $a$ :

$$\frac{1}{f} \left( \partial_x^2 + \partial_y^2 \right) f = \text{constant} = -a^2. \quad (50)$$

Then, our first equation (44) becomes, via Eqns. 47 and 50

$$\begin{aligned}
s \left[ D^2 \bar{w} - a^2 \bar{w} \right] f &= -\alpha g a^2 f \theta + \nu \left[ \left( D^2 - a^2 \right)^2 \bar{w} f \right] \\
s \left[ D^2 - a^2 \right] \bar{w} &= -\alpha g a^2 \theta + \nu \left[ D^2 - a^2 \right]^2 \bar{w} \\
\left[ D^2 - a^2 \right] \left[ D^2 - a^2 - \frac{s}{\nu} \right] \bar{w} &= \alpha g a^2 \frac{\theta}{\nu}
\end{aligned} \tag{51}$$

and our second equation (45) becomes, via Eqns. 48 and 50

$$\begin{aligned}
\kappa \theta'' - a^2 \kappa \theta - s \theta &= -\beta \bar{w} \\
\left( D^2 - a^2 - \frac{s}{\kappa} \right) \theta &= -\frac{\beta}{\kappa} \bar{w}
\end{aligned} \tag{52}$$

In these equations,  $D$  is the differential operator:  $D\bar{w} = d\bar{w}/dz$ ,  $D^2\bar{w} = d^2\bar{w}/dz^2$ , etc. Eqns. 51 and 52 comprise two equations in the two unknown functions  $\bar{w}$  and  $\theta$ . Finally, they can be combined by eliminating  $\theta$  to give just one equation for  $\bar{w}$ :

$$\left[ D^2 - a^2 - \frac{s}{\kappa} \right] \left[ D^2 - a^2 \right] \left[ D^2 - a^2 - \frac{s}{\nu} \right] \bar{w} = -\frac{a^2 \alpha g \beta}{\kappa \nu} \bar{w}. \tag{53}$$

### 2.5.2 Adimensionalisation. The Rayleigh and Prandtl numbers

So far we have kept the equations in dimensional form. We now convert to adimensional form. To do so, we choose to measure temperatures in units of the temperature difference  $\beta d$ ; lengths in units of the gap width  $d$ ; and times in units of  $d^2/\kappa$ , which can be interpreted as the time that any thermal fluctuation takes to diffuse across the gap. Accordingly, in Eqn. 53 we define the dimensionless quantities

$$\hat{z} = \frac{z}{d}, \quad \hat{D} = dD, \quad \hat{a} = da, \quad \hat{s} = \frac{sd^2}{\kappa} \text{ and } \hat{w} = \frac{d}{\kappa} \bar{w}. \tag{54}$$

Substituting these into Eqn. 53 gives (after a bit of algebra)

$$\left[ \hat{D}^2 - \hat{a}^2 - \hat{s} \right] \left[ \hat{D}^2 - \hat{a}^2 \right] \left[ \hat{D}^2 - \hat{a}^2 - \frac{\hat{s}\kappa}{\nu} \right] \hat{w} = -\frac{\hat{a}^2 \alpha d^4 g \beta}{\kappa \nu} \hat{w}. \tag{55}$$

Dropping the hats for clarity we get finally the adimensional form

$$\left[ D^2 - a^2 - s \right] \left[ D^2 - a^2 \right] \left[ D^2 - a^2 - \frac{s}{\text{P}} \right] \bar{w} = -a^2 \text{R} \bar{w}. \tag{56}$$

This equation contains two newly defined quantities, P and R.

$$\text{The Prandtl number } \text{P} = \frac{\nu}{\kappa} \tag{57}$$

is the ratio of the fluid's kinematic viscosity, which prescribes the rate of momentum diffusion, to its thermal diffusivity, which prescribes the rate of heat diffusion. It is thus an intrinsic property of a given fluid.

$$\text{The Rayleigh number } R = \frac{\alpha d^4 g \beta}{\kappa \nu}, \quad (58)$$

which contains the applied thermal gradient  $\beta$ , will be used as the control parameter to be varied externally by the experimenter. As shown quantitatively in Sec. 2.5.5 below, the base flow is stable for small  $R$ , and unstable with respect to convection rolls for large  $R$ . Qualitatively this result can be anticipated by considering the relative importance of stabilising and destabilising influences in the problem, as we now show.

**Destabilising** — A positive temperature fluctuation  $\Delta T$  in any fluid element leads to a local decrease in density, which tends to accelerate the element upwards. Per unit volume we have

$$\rho \alpha g \Delta T = \text{force} = \rho \times \text{acceleration} = \rho \frac{d}{\tau_b^2}, \quad (59)$$

which defines a characteristic time  $\tau_b$  for the element to rise a distance  $d$  (gap width).

**Stabilising** — Fluctuations tend to be damped back to uniformity by viscous friction and heat diffusion. Characteristic timescales for these processes are

$$\tau_\nu = \frac{d^2}{\nu} \quad \text{for diffusion of momentum via viscous friction}, \quad (60)$$

and

$$\tau_\kappa = \frac{d^2}{\kappa} \quad \text{for heat diffusion}. \quad (61)$$

The Rayleigh number can then be understood as  $R = \tau_\nu \tau_\kappa / \tau_b^2$ . At low Rayleigh number (small  $\tau_\nu, \tau_\kappa$ ), the damping processes happen quickly enough to counteract the buoyant rising instability of a warm element. By contrast, at high Rayleigh number (small  $\tau_b$ ) the buoyancy is strong enough to overturn the layer. The threshold for convection is thus expected to lie at some intermediate value  $R = R_c$ . The quantitative analysis in Sec. 2.5.5 will confirm this intuition, and result in a concrete expression for  $R_c$ .

### 2.5.3 Boundary conditions

In Eqns. 25 to 27 above, we gave the conditions obeyed by the fields  $\mathbf{u}(\mathbf{x}, t)$  and  $T(\mathbf{x}, t)$  at the boundaries. We now need to convert these into equivalent conditions for the quantity  $\bar{w}$  at the adimensionalised boundaries  $z = 0, 1$ . Eqn. 53 is a **sixth order equation**, and so requires **three conditions at each boundary** in order to fully specify its solution. We consider fixed-fixed and free-free boundaries in turn.

**Two fixed walls** — For the velocity field we have no permeation

$$\bar{w}(z) = 0 \quad \text{at } z = 0, 1. \quad (62)$$

No slip gives  $\tilde{u} = \tilde{v} = 0 \quad \forall x, y$  on  $z = 0, d$ . Together with continuity  $\partial_x \tilde{u} + \partial_y \tilde{v} + \partial_z \tilde{w} = 0$  (Eqn. 36) this gives

$$\frac{d\bar{w}}{dz} = 0 \quad \text{at } z = 0, 1. \quad (63)$$

The condition of constant applied temperature along each boundary means that the temperature perturbation is zero at each wall:

$$\theta = 0 \quad \text{at } z = 0, 1. \quad (64)$$

This can be converted to an equivalent condition on  $\bar{w}$ , though we do not pursue this.

**Two free fluid surfaces** — The boundary conditions on a free surface are that the perturbations of the components of stress are zero. This can be shown to give

$$\bar{w} = \frac{d^2\bar{w}}{dz^2} = 0 \quad \text{at } z = 0, 1. \quad (65)$$

The condition of constant applied temperature at each boundary means that the temperature perturbation is zero at each wall. From this we find that  $\theta = 0$  at  $z = 0, 1$ . Substituting this, together with (65) into Eqn. 51, we get

$$\frac{d^4\bar{w}}{dz^4} = 0 \quad \text{at } z = 0, 1. \quad (66)$$

#### 2.5.4 The story so far

Our progress so far can be summarised as follows:

- The base state has no flow  $\mathbf{u}_B = 0$  and a vertical temperature profile  $T_B = T_0 - \beta z$ .
- The perturbed state has the general form

$$\mathbf{u} = \mathbf{u}_B + \delta\tilde{\mathbf{u}}(\mathbf{x}, t) \quad , \quad T = T_B(z) + \delta\tilde{T}(\mathbf{x}, t) \quad , \quad p = p_B(z) + \delta\tilde{p}(\mathbf{x}, t). \quad (67)$$

where the perturbations  $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{T}, \tilde{p}$  obey the linearised equations 36 to 40.  $\tilde{u}, \tilde{v}, \tilde{p}$  can be eliminated to leave Eqns. 44 and 45 for  $\tilde{w}$  and  $\tilde{T}$ .

- Separable normal mode solutions are of the form

$$\tilde{w}(x, y, z, t) = f(x, y)\bar{w}(z) \exp(st) \quad \text{and} \quad \tilde{T}(x, y, z, t) = f(x, y)\theta(z) \exp(st) \quad (68)$$

in which  $f$  obeys

$$\frac{1}{f} \left( \partial_x^2 + \partial_y^2 \right) f = \text{constant} = -a^2. \quad (69)$$

$\bar{w}(z)$  and  $s$  can then be shown to obey the adimensionalised equation

$$\left[ D^2 - a^2 - s \right] \left[ D^2 - a^2 \right] \left[ D^2 - a^2 - \frac{s}{P} \right] \bar{w} = -a^2 R \bar{w}, \quad (70)$$

from which  $\tilde{T}$  has now also been eliminated.

- Assuming (unrealistically) that both fluid surfaces are open to the air, the free-free boundary conditions can be shown to be (Eqns. 65, 66)

$$\bar{w} = \bar{w}^{(ii)} = \bar{w}^{(iv)} = 0 \quad \text{on } z = 0, 1. \quad (71)$$

The case of rigid bounding walls is more difficult to solve, and not pursued here.

### 2.5.5 Rayleigh's solution for free-free boundaries

We now want to solve for the perturbation

$$\tilde{w}(x, y, z, t) = f(x, y)\bar{w}(z)\exp(st) \quad (72)$$

in the case of free-free boundaries. To do so, we solve the eigenvalue problem (70) for  $s$  and  $\bar{w}$  subject to the free-free boundary conditions (71). Eqn. 69 can then be used to find  $f(x, y)$ . As discussed above, the condition for linear instability is that the real part of the eigenvalue is positive:  $s_r > 0$ . Small perturbations then grow in time via the exponential factor in Eqn. 72. The corresponding function  $\bar{w}(z)f(x, y)$  determines the spatial dependence of the growing perturbation.

Rayleigh noted that if

$$\bar{w} = \bar{w}^{(ii)} = \bar{w}^{(iv)} = 0 \quad \text{on} \quad z = 0, 1, \quad (73)$$

then Eqn. 70 means that also

$$\bar{w}^{(vi)} = 0 \quad \text{on} \quad z = 0, 1. \quad (74)$$

Similarly

$$\bar{w}^{(2n)} = 0 \quad \text{on} \quad z = 0, 1 \quad \text{for} \quad n = 1, 2, \dots \quad (75)$$

This led Rayleigh to seek solutions

$$\bar{w} = A \sin(n\pi z) \quad \text{for} \quad n = 1, 2, \dots \quad (76)$$

Eqn. 70 is then satisfied if

$$\left(-n^2\pi^2 - a^2 - s\right) \left(-n^2\pi^2 - a^2\right) \left(-n^2\pi^2 - a^2 - \frac{s}{P}\right) A = -a^2RA. \quad (77)$$

Recall that  $P$  is the (constant) Prandtl number.  $R$  is the Rayleigh number, used as the control parameter.  $n$  is the mode number in the  $z$  direction.  $a$  is effectively a mode number in the  $x, y$  plane (Eqn. 50).  $s$  is the eigenvalue, which we want to calculate.

For convenience we define

$$\lambda = n^2\pi^2 + a^2 \quad (78)$$

and rewrite (77) as

$$(s + \lambda) \left(\frac{s}{P} + \lambda\right) = \frac{a^2R}{\lambda}, \quad (79)$$

and then

$$s^2 + \lambda(P + 1)s + \left(\lambda^2P - \frac{a^2RP}{\lambda}\right) = 0. \quad (80)$$

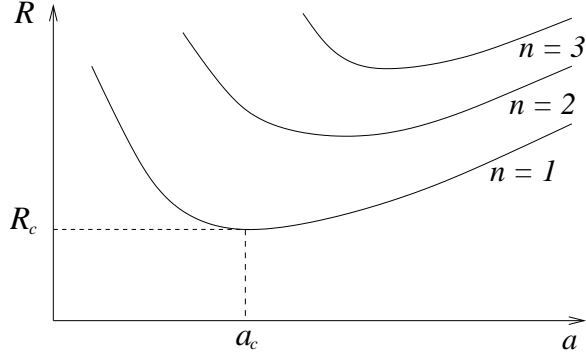
As a quadratic in  $s$ , this is easily solved to give

$$s = -\frac{\lambda(P + 1)}{2} \pm \frac{1}{2}\sqrt{\lambda^2(P + 1)^2 + 4P\left(\frac{a^2R}{\lambda} - \lambda^2\right)}. \quad (81)$$

Supposing that  $s$  is real, the condition for instability  $s > 0$  for given mode numbers  $n, a$  can then be shown to be

$$\frac{a^2R}{\lambda} - \lambda^2 > 0 \quad (82)$$





and so

$$R > R_n(a) = \frac{\lambda^3}{a^2} = \frac{(n^2\pi^2 + a^2)^3}{a^2}. \quad (83)$$

The condition for neutral stability  $R = R_n(a)$  is sketched above for  $n = 1, 2, 3$ . For small Rayleigh number  $R$  all modes  $n = 1, 2, 3 \dots$  are stable. As the Rayleigh number increases, the basic state (no convection) first becomes unstable at  $R = R_c$  to a convective state prescribed by the eigenmode with mode number  $n = 1$  in the vertical direction and  $a = a_c$  in the  $x - y$  plane.  $R_c$  and  $a_c$  are found by minimising  $R_1(a)$  with respect to  $a$ . Do this as an exercise:

$$\frac{dR_1}{da} = 0 \quad \text{gives} \quad a_c^2 = \frac{\pi^2}{2}. \quad (84)$$

The instability thus first sets in at the critical Rayleigh number

$$R_c = R_1(a_c) = \frac{\left(\pi^2 + \frac{\pi^2}{2}\right)^3}{\left(\frac{\pi^2}{2}\right)} = \frac{27\pi^2}{4} \approx 657.5 \quad (85)$$

The spatial dependence of this most unstable mode is as follows. In the  $z$  direction we have  $\bar{w} = A \sin(\pi z)$  (Eqn. 76). In the  $x - y$  plane, we must solve Eqn. 50 with  $a = a_c$ :

$$\nabla^2 f + a_c^2 f = 0. \quad (86)$$

This has many possible solutions, the simplest of which is a roll state with no  $y$  dependence and

$$f(x, y) = \cos\left(\frac{2\pi x}{L_x}\right). \quad (87)$$

Substituting this into Eqn. 86 gives  $\frac{4\pi^2}{L_x^2} = a_c^2$  and so a wavelength  $L_x = \sqrt{8}$ .

