4.2 The Falkner-Skan equation

For the case of a flat plate, we have just shown that the boundary layer solution is self-similar in the sense that two profiles $u(x,y)$ at different values of x differ only by a scale factor in y. This simplified the analysis considerably, because the momentum equation reduced to an ordinary differential equation in the single similarity variable η. What about more complicated geometries? In this section, we derive the general conditions under which self-similar solutions exist. As usual we work in terms of the stream function Ψ so that the continuity equation is automatically satisfied.

Consider again the boundary layer transformations 65, which tell us that the basic scales of the y coordinate and the stream function are

$$y' = Re^{1/2}\frac{y}{L} \quad \text{and} \quad \Psi(x,y) = \frac{LU}{Re^{1/2}}\Psi'(x',y'). \tag{106}$$

For a flat plate, we argued that we must actually replace the overall length of the object L by the present distance travelled along it, x. This gave (in an unusual format)

$$\eta = \frac{Re^{1/2}}{\sqrt{2x/L}}\frac{y}{L} \quad \text{and} \quad \Psi = \frac{LU\sqrt{2x/L}}{Re^{1/2}}f(\eta), \tag{107}$$

in which the scaled stream function depends on space only through the single similarity variable η.

To extend this analysis to more general geometries, we recall that the only way the geometry enters the BL equations is through its influence on $u_e(x)$. For a flat plate just considered, $u_e(x) = U =$ constant. More generally, $u_e = u_e(x)$. In what follows, we aim to find the most general form of $u_e(x)$ that still admits self-similar solutions. To do so, we postulate a generalised similarity transformation

$$\eta = \frac{Re^{1/2}}{h(x)}\frac{y}{L} \quad \text{and} \quad \Psi = \frac{Lu_e(x)h(x)}{Re^{1/2}}f(\eta). \tag{108}$$

where $h(x)$ is an unknown, non-dimensional function of x representing the dependence of η on x. For $h(x) = \sqrt{2x/L}$ and $u_e(x) = U$ we recover the flat-plate variables 107. (Notice that in this more general case we are rescaling both y and u by functions of x. For the flat plate, with u_e constant, we only needed to rescale y.) For convenience, we define a second non-dimensional variable,

$$\xi = \frac{x}{L}. \tag{109}$$

We now change coordinates from (x,y) to (ξ,η) in the momentum BL equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2}. \tag{110}$$

To do so, we start by calculating transformed expressions for $\partial/\partial x$ and $\partial/\partial y$:

$$\frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial \xi} - \eta\frac{h'(x)}{h(x)}\frac{\partial}{\partial \eta}, \quad \text{and} \quad \frac{\partial}{\partial y} = \frac{Re^{1/2}}{Lh(x)}\frac{\partial}{\partial \eta}. \tag{111}$$

The velocity components u and v are then

$$u = \frac{\partial \Psi}{\partial y} = u_e(x)f'(\eta), \tag{112}$$

$$-v = \frac{\partial\Psi}{\partial x} = \frac{1}{Re^{1/2}}\left\{\frac{d}{d\xi}(u_e(x)h(x))f(\eta) - Lu_e(x)h'(x)\eta f'(\eta)\right\}. \tag{113}$$

Using these, we now calculate all the component terms of the momentum equation 110.

- The viscous term is

$$\begin{aligned}\nu\frac{\partial^2 u}{\partial y^2} &= \nu\frac{Re^{1/2}}{Lh}\frac{\partial}{\partial\eta}\left(\frac{Re^{1/2}}{Lh}\frac{\partial}{\partial\eta}(u_e f')\right)\\ &= \frac{U}{L}\frac{u_e}{h^2}f'''.\end{aligned} \tag{114}$$

- The advection terms are

$$\begin{aligned}u\frac{\partial u}{\partial x} &= u_e f'\left\{\frac{1}{L}\frac{\partial}{\partial\xi} - \eta\frac{h'}{h}\frac{\partial}{\partial\eta}\right\}(u_e f')\\ &= \frac{u_e}{L}\frac{du_e}{d\xi}f'^2 - u_e^2\frac{h'}{h}\eta f' f'',\end{aligned}$$

  and

$$\begin{aligned}v\frac{\partial u}{\partial y} &= \left\{-\frac{1}{Re^{1/2}}\frac{d}{d\xi}(u_e h)f + \frac{L}{Re^{1/2}}u_e h'\eta f'\right\}\frac{Re^{1/2}}{Lh}\frac{\partial}{\partial\eta}(u_e f')\\ &= -\frac{u_e}{Lh}\frac{d}{d\xi}(u_e h)ff'' + u_e^2\frac{h'}{h}\eta f' f''.\end{aligned}$$

- The pressure term, expressed as usual in terms of the exterior slipping velocity, is

$$u_e(x)\frac{du_e(x)}{dx} = \frac{u_e(x)}{L}\frac{du_e(x)}{d\xi}. \tag{115}$$

Putting all these together, we get the transformed momentum equation

$$\frac{U}{L}\frac{u_e}{h^2}f''' + \frac{u_e}{Lh}\frac{d}{d\xi}(u_e h)ff'' + \frac{u_e}{L}\frac{du_e}{d\xi}\{1 - f'^2\} = 0. \tag{116}$$

Multiplying across by Lh^2/Uu_e we get

$$f''' + \frac{h}{U}\frac{d}{d\xi}(u_e h)ff'' + \frac{h^2}{U}\frac{du_e}{d\xi}\{1 - f'^2\} = 0. \tag{117}$$

We rewrite this as

$$f''' + \alpha ff'' + \beta(1 - f'^2) = 0, \tag{118}$$

in which (using $d/d\xi = Ld/dx$ to revert back $\xi \to x$)

$$\alpha = \frac{Lh}{U}\frac{d}{dx}(u_e h) \quad \text{and} \quad \beta = \frac{Lh^2}{U}\frac{du_e}{dx}. \tag{119}$$

The boundary conditions are (as usual)

$$f(0) = f'(0) = 0 \tag{120}$$

and

$$f'(\infty) = 1. \tag{121}$$

Recall now the arguments of the various functions in Eqn. 117: $f = f(\eta)$, $u_e = u_e(x)$, $h = h(x)$ and so (a priori at least) $\alpha = \alpha(x)$ and $\beta = \beta(x)$. For a self-similar solution to exist, α and β must actually be constants, independent of x (or equivalently of ξ). By forming appropriate linear combinations of α and β, and integrating the expressions thus obtained, we can calculate the forms that $u_e(x)$ and $h(x)$ must take in order for this to be the case. (We do not give the details here: they can be found in Schlichting, 'Boundary Layer Theory', McGraw Hill, chapter VIII b.) These are

$$\frac{u_e(x)}{U} = K^{\frac{2}{2\alpha-\beta}} \left[(2\alpha - \beta)\frac{x}{L}\right]^{\frac{\beta}{2\alpha-\beta}}, \tag{122}$$

and

$$h(x) = \sqrt{(2\alpha - \beta)\frac{x}{L}\frac{U}{u_e}}, \tag{123}$$

where K is an integration constant. The requirement $2\alpha - \beta \neq 0$ is imposed. We do not consider the case $2\alpha - \beta = 0$. As can be seen, the overall result is independent of any common factor in α and β, as this can simply be absorbed into the definition of h. As long as $\alpha = 0$, therefore, we are free to choose $\alpha = 1$. For convenience we also introduce a new constant

$$m = \frac{\beta}{2-\beta}, \quad \text{so that} \quad \beta = \frac{2m}{m+1}. \tag{124}$$

Using 122, 123 and 124 we get

$$\frac{u_e(x)}{U} = K^{m+1}\left(\frac{2}{m+1}\frac{x}{L}\right)^m, \tag{125}$$

and

$$h(x) = \sqrt{\frac{2}{m+1}\frac{x}{L}\frac{U}{u_e}}. \tag{126}$$

Substituting $h(x)$ into Eqn. 108, we obtain the final form of the similarity variable

$$\eta = y\sqrt{\frac{m+1}{2}\frac{u_e}{\nu x}} = Re_x^{1/2}\,\frac{y}{x}\sqrt{\frac{m+1}{2}} \quad \text{with} \quad Re_x = \frac{u_e x}{\nu}. \tag{127}$$

Summary

For an exterior flow $u_e(x) \propto x^m$ we have the similarity variable η given by Eqn. 127. The scaled stream function $f(\eta)$ then obeys the ordinary differential equation

$$f''' + ff'' + \beta(1 - f'^2) = 0 \tag{128}$$

with β constant. The boundary conditions are

$$f'(0) = f(0) = 0, \quad f'(\infty) = 1. \tag{129}$$

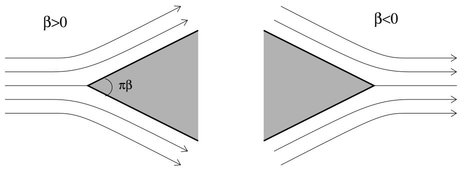
This equation is known as the **Falkner-Skan** equation.

4.2.1 Physical interpretation

Looking back at Eqn. 125, we see that self-similar solutions of the 2D boundary layer equations exist if the exterior inviscid flow is a power law

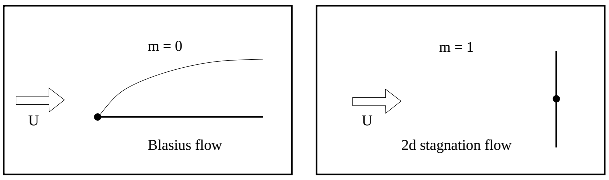
$$u_e(x) = Cx^m, \qquad C \text{ constant.} \tag{130}$$

It turns out (not shown here) that this is the profile of flow past a wedge of angle $\pi\beta$.



4.2.2 Special cases

- $m = 0$: Blasius flow over a flat plate, Sec. 4.1.
- $m = 1$: 2D stagnation flow, *e.g.*, near the nose of a cylinder (problem sheet 3).



4.2.3 Numerical results

In accordance with the discussion of Sec. 4.1, the Falkner-Skan equation must be solved numerically. The velocity profile is shown in Fig. 10 for different values of m. Note that it develops an inflexion point as m (and hence also β) becomes negative. This is closely related to the phenomenon of boundary layer separation, to which we now turn.

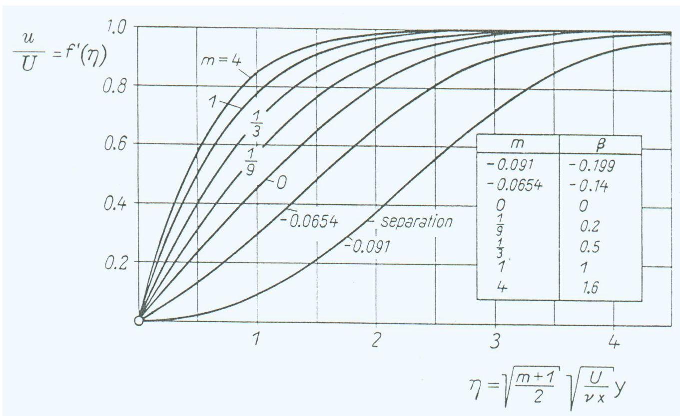


Figure 10: Falkner-Skan profiles. (Schlichting, 'Boundary Layer Theory', McGraw Hill).