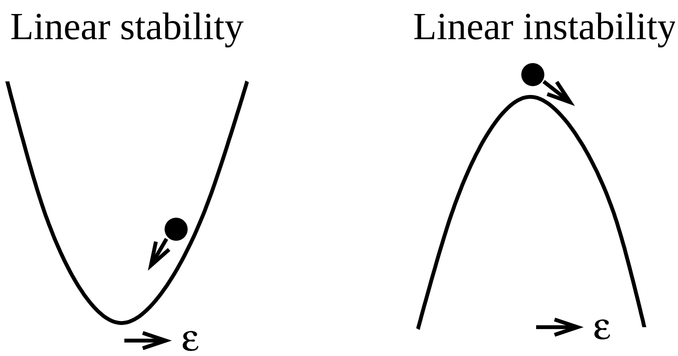
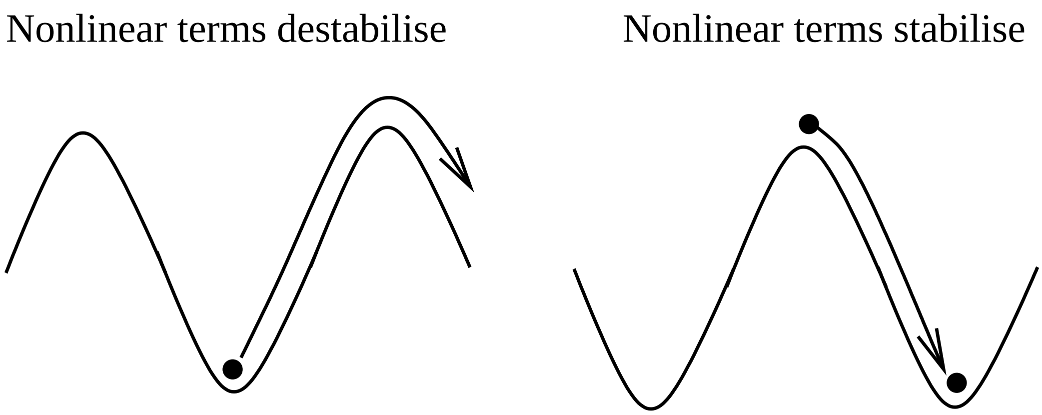
5 Introduction

In part I of the course, we considered perturbations of small amplitude $\epsilon \ll 1$ to an initial base state. (Note the change of notation. In part I we denoted the amplitude by δ. In part II we will reserve δ to instead denote a control parameter, introduced below.) We distinguished the case of linear instability, in which perturbations grow in time, from that of linear stability, in which they decay. Using the "particle in a well" analogy introduced in part I of the course, these two scenarios are sketched as follows.

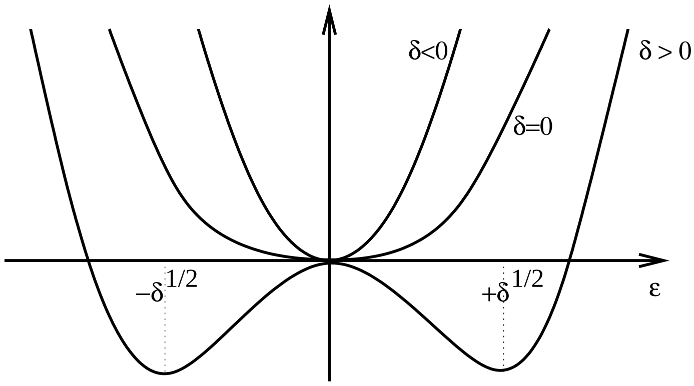


In the right hand sketch, a linear stability analysis tells us that a small perturbation will initially grow in time. At later times, however, once the perturbation has grown to attain a larger amplitude, the assumption of linearity breaks down. The linearised equations no longer apply, and nonlinear effects become important. For example, nonlinear effects may eventually restabilise the system, as shown in the right hand sketch below. The top sketch is thus just the small-ϵ part of the bottom one.



Conversely, a system that is linearly *stable* with respect to small perturbations might experience nonlinear effects that tend to destabilise it. (Left hand sketch.) A sufficiently large perturbation could then activate the system out of the original base state, even though that state would be predicted to be stable with respect to infinitesimal perturbations at the level of a linear analysis.

For the rest of the course, our focus will be on nonlinear effects. A key concept will be that of local bifurcation, in which a base state switches from being linearly stable to linearly unstable (or vice versa) as an externally applied control parameter δ is smoothly varied. Of course, this concept has already been encountered in part I of the course. In the case of Bénard convection, for example, a non-convecting state becomes linearly unstable with respect to the formation of convection rolls above a critical applied temperature gradient. The new focus here will be on the way in which nonlinear terms can (for example) restabilise the system beyond the threshold of instability, in a new nonlinear state.



In this example, the state at $\epsilon = 0$ is linearly stable for $\delta < 0$. In the "bifurcation diagram" below, this is represented by the solid horizontal line. For $\delta > 0$ the state at $\epsilon = 0$ becomes linearly unstable (dashed horizontal line below). Two new stable states at then emerge at $\epsilon = \pm\sqrt{\delta}$. Depending on initial conditions, the system will settle to one of these at long times.