6 A review of linear stability analysis

Before turning to our study of nonlinear effects in Sec. 7 below, first we will review what we already know about linear stability analysis from part I of the course. We do this in the context of a partial differential equation called the Eckhaus equation. (This was originally derived in the context of convection in a porous medium. We use it here because it is simpler than our familiar Navier-Stokes equation.) This section will also serve as an introduction to the Eckhaus equation, in anticipation of studying its nonlinear dynamics later on. We follow the usual procedure of a linear stability analysis, which was explained at length in part I of the course.

6.1 Governing equations and boundary conditions

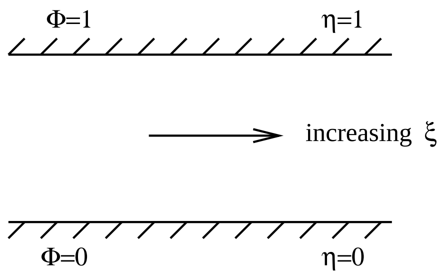
The Eckhaus equation is as follows:

$$\frac{1}{R}\left[\Phi_{\eta\eta} + \Phi_{\xi\xi}\right] - \Phi_{\xi\xi\xi\xi} - \Phi_t = \Phi_\eta \Phi_{\xi\xi}. \tag{1}$$

It is nonlinear, because of the product on the RHS.

- t is time,
- ξ and η are spatial coordinates,
- and R is a Reynolds-number-like control parameter.

We consider the domain sketched below, with Φ bounded as $|\xi| \to \infty$.



The boundary conditions are as follows

- $\Phi = 0 \quad \text{on} \quad \eta = 0 \quad \forall\, \xi, t.$
- $\Phi = 1 \quad \text{on} \quad \eta = 1 \quad \forall\, \xi, t.$

6.2 The base state

There is an obvious basic state

$$\Phi_{\mathrm{B}} = \eta, \tag{2}$$

which satisfies the Eckhaus equation and the given boundary conditions.

6.3 Add a small perturbation

We now add a small perturbation:

$$\begin{aligned} \Phi &= \Phi_{\mathrm{B}}(\eta) + \phi(\xi, \eta, t) \\ &= \eta + \phi. \end{aligned} \tag{3}$$

6.4 Linearise the equations

On linearising about the basic state, a bit of care is needed with the nonlinear term:

$$\begin{aligned}\Phi_\eta \Phi_{\xi\xi} &= (\eta+\phi)_\eta(\eta+\phi)_{\xi\xi} \\ &= (1+\phi_\eta)\phi_{\xi\xi} \\ &\approx \phi_{\xi\xi} \quad \text{for} \quad |\phi| \ll 1. \end{aligned} \tag{4}$$

The linearised equation is thus

$$\frac{1}{R}\left[\phi_{\eta\eta}+\phi_{\xi\xi}\right]-\phi_{\xi\xi\xi\xi}-\phi_t=\phi_{\xi\xi}, \tag{5}$$

with boundary conditions

- $\phi = 0 \quad \text{on} \quad \eta = 0, 1 \quad \forall\, \xi, t.$

6.5 Solve the linearised equations using normal modes

We now look for normal modes in the form

$$\phi = \cos(k\xi)F(\eta)\exp(\sigma t), \tag{6}$$

i.e., a solution that is periodic in ξ, with an exponential dependence on time. As usual, as $t \to \infty$ we have

- exponential growth – linear instability – if $\Re(\sigma) > 0$; and
- exponential decay – linear stability – if $\Re(\sigma) < 0$.

Substituting (6) into (5), we obtain an equation for $F(\eta)$ as follows:

$$\frac{1}{R}\left[F'' - k^2F\right] - (-k^2)^2F - \sigma F = -k^2F. \tag{7}$$

(Check this an exercise.) Tidying up, we get:

$$F'' - k^2F - k^4RF - R\sigma F + k^2RF = 0, \tag{8}$$

and so finally

$$F'' + F\left[R(k^2 - k^4 - \sigma) - k^2\right] = 0, \tag{9}$$

with boundary conditions

- $F = 0 \quad \text{on} \quad \eta = 0, 1.$

Because (9) is of the general form $F'' + c^2F = 0$, it has solutions

$$F(\eta) = A_n \sin(n\pi\eta) \quad \text{for} \quad n = 1, 2, 3\cdots, \tag{10}$$

with the corresponding eigenrelation

$$n^2\pi^2 = R(k^2 - k^4 - \sigma) - k^2. \tag{11}$$

To determine whether the basic state Φ_B is linearly stable or unstable, we obviously need an expression for σ. This follows easily from (11):

$$\sigma = k^2 - k^4 - R^{-1}\left[n^2\pi^2 + k^2\right]. \tag{12}$$

So in this example, σ is real and we have

- linear instability if $\sigma > 0$,
- linear stability if $\sigma < 0$,
- neutral stability if $\sigma = 0$.

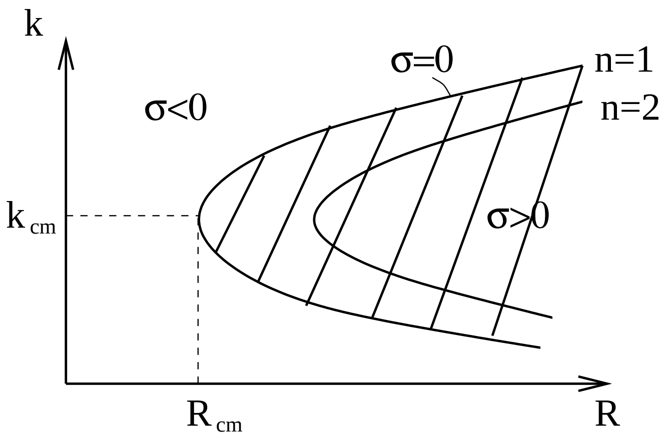
So the neutral stability condition ($\sigma = 0$) is given by

$$k^2 - k^4 - R^{-1}\left[n^2\pi^2 + k^2\right] = 0. \tag{13}$$

For any mode number n and wavevector k, this gives the critical value of R as

$$R_{\mathrm{c}} = \frac{n^2\pi^2 + k^2}{k^2 - k^4}. \tag{14}$$

These lines of neutral stability are sketched below for mode numbers $n = 1, 2$.



As the control parameter R is increased, the base state first becomes linearly unstable at $R = R_{\mathrm{cm}}$ with respect to perturbations with mode number $n = 1$ and wavevector $k = k_{\mathrm{cm}}$. To find $R_{\mathrm{cm}}, k_{\mathrm{cm}}$, we simply need to find the point at which $dR_{\mathrm{c}}/dk = 0$ for this mode, $n = 1$. (See the examples sheet. We find $R_{\mathrm{cm}} \approx 40$ and $k_{\mathrm{cm}} \approx 0.7$.) From (6) and (10), we recall that these perturbations take the form:

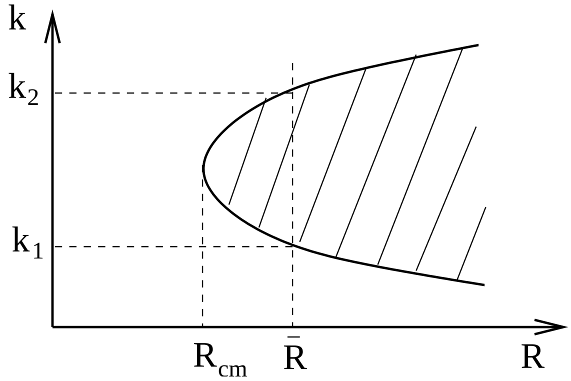
$$\sin(\pi\eta)\cos(k_{\mathrm{cm}}\xi). \tag{15}$$

Having reviewed the procedure involved in a linear stability analysis, we will now undertake a critique of it, before proceeding to our study of nonlinear effects.

6.6 Criticisms of linear stability analysis

Linear stability analysis may be criticised as follows.

- By neglecting nonlinear terms, which describe the interaction of the perturbations with themselves ($\phi_\eta \phi_{\xi\xi}$ in the case of the Eckhaus equation), linear analysis is restricted to the regime in which the amplitude of perturbations remains very small.
- Because of this, in the linearly unstable regime a linear analysis cannot tell us about the fate of perturbations at long times, once they have grown to have an amplitude that is not infinitesimal. It therefore cannot tell us about the ultimate structure of the flow in the limit $t \to \infty$. Recall the example of the Eckhaus equation, above. The linear analysis tells us that for $R > R_{\text{cm}} \approx 40$ we will not see the basic state $\Phi = \Phi_{\text{B}}$, because it is linearly unstable. It furthermore *suggests* that some appearance of a new state with $\xi-$wavelength$\approx 2\pi/k_{\text{cm}}$ may be measurable in an experiment, but that is all.
- At any fixed $R = \bar{R} > R_{\text{cm}}$, there is actually a band of unstable wavevectors $k_1 < k < k_2$ that are unstable, and so may be expected to appear together. In a linear analysis, we typically only consider the single wavevector for which the growth rate σ is largest.



- Again recalling the example of the Eckhaus equation, the linear analysis tells us that the simple base state $\Phi_{\text{B}} = \eta$ is *linearly* stable up to the critical value of the control parameter $R = R_{\text{cm}} \approx 40$. However, it cannot tell us whether this base state will actually be observed experimentally for all value of R up to R_{cm}. This is because nonlinear effects might (or might not) cause destabilisation, even for values $R < R_{\text{cm}}$. Recall the bottom left sketch on page 2. We would need study the model's nonlinear dynamics, and know about the level of background noise in the system, to determine this.